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1 Final Good and Processed Good Sectors

1.1 Final Good Sector

The final good sector is perfectly competitive with Cobb-Douglas production:

\[ Y = X_1^\epsilon X_2^{1-\epsilon} \]  

(1)

Final good producers maximize profit by choosing the amount of processed good given the prices of those goods. Let \( Y \) be the numeraire. The Lagrangean for final good producers is

\[ \pi_Y = Y - P_{X_1} X_1 - P_{X_2} X_2 \]  

(2)

The first order conditions are

\[ \frac{\partial \pi_Y}{\partial X_1} = \epsilon X_1^{\epsilon-1} X_2^{1-\epsilon} - P_{X_1} = 0 \]  

(3)

\[ \frac{\partial \pi_Y}{\partial X_2} = (1 - \epsilon) X_1^\epsilon X_2^{-\epsilon} - P_{X_2} = 0 \]  

(4)

from which we obtain the demand functions

\[ X_1 = \left( \frac{\epsilon}{P_{X_1}} \right)^{\frac{1}{1-\epsilon}} X_2 \]  

(5)

\[ X_2 = \left( \frac{1 - \epsilon}{P_{X_2}} \right)^{\frac{1}{\epsilon}} X_1 \]  

(6)

The inverse demand functions for \( X_i \)’s are

\[ P_{X_1} = \epsilon X_1^{\epsilon-1} X_2^{1-\epsilon} \]  

(7)

\[ P_{X_2} = (1 - \epsilon) X_1^\epsilon X_2^{-\epsilon} \]  

(8)

which together yield

\[ \frac{P_{X_1}}{P_{X_2}} = \frac{\epsilon}{1 - \epsilon} \left( \frac{X_1}{X_2} \right)^{1-\epsilon} \]  

(9)

The competitive final-good producer pays compensation \( \epsilon Y \) and \( (1 - \epsilon) Y \) to the processed-good 1 and processed-good 2. So we get \( \epsilon Y = P_{X_1} X_1 \) and \( (1 - \epsilon) Y = P_{X_2} X_2 \).

1.2 Processed Good Sector

The processed good sector also is perfectly competitive. The production functions for the two processed good industries \( X_1 \) and \( X_2 \) are

\[ X_1 = \int_0^{N_1} G_{1j} \left( Z_{1j}^\lambda Z_1^{\delta} Z_2^{1-(\delta+\gamma)} l_{1j} \right)^{1-\lambda} dj, \quad 0 < \lambda, \delta < 1 \]  

\[ X_2 = \int_0^{N_2} G_{2j} \left( Z_{2j}^\lambda Z_2^{\delta} Z_1^{1-(\delta+\gamma)} l_{2j} \right)^{1-\lambda} dj, \quad 0 < \lambda, \delta < 1 \]  

(10)

where \( Z_1 = (1/N_1) \int_0^{N_1} Z_{1j} dj \), and \( Z_2 = (1/N_1) \int_0^{N_2} Z_{2j} dj \) are the average qualities of the intermediate goods \( G_{1j} \) and \( G_{2j} \). We assume labor moves freely across firms and industries, so wages must be equal everywhere. Taking the prices of intermediate goods and the wage as given, processed goods producers in the first (i.e., \( X_1 \)) industry choose the amounts of the \( G_{1j} \) and labor \( l_{1j} \) to maximize profit:

\[ \pi_{X_1} = P_{X_1} X_1 - \int_0^{N_1} P_{G_{1j}} G_{1j} dj - \int_0^{N_1} w l_{1j} dj \]
The first order conditions are
\[
\frac{\partial \pi_{X_1}}{\partial G_{ij}} = P_{X_1}(\frac{\partial X_1}{\partial G_{ij}}) - P_{G_{ij}} \\
= P_{X_1} \lambda G_{ij}^{\lambda-1}(Z_1^\delta Z_2^{1-(\delta+\gamma)} l_{1j})^{1-\lambda} - P_{G_{ij}} \\
= 0 \\
\frac{\partial \pi_{X_1}}{\partial l_{1j}} = P_{X_1}(\frac{\partial X_1}{\partial l_{1j}}) - w_{1j} \\
= P_{X_1}(1-\lambda)Z_1^\delta Z_2^{1-(\delta+\gamma)} l_{1j} - w_{1j} \\
= 0
\]
from which we obtain
\[
G_{1j} = \left[\frac{\lambda P_{X_1}}{P_{G_{1j}}}\right]^{\frac{1}{1-\lambda}} Z_1^\delta Z_2^{1-(\delta+\gamma)} l_{1j} \tag{11}
\]
\[
l_{1j} = (P_{X_1} \frac{1-\lambda}{w})^{\frac{1}{\lambda}} G_{1j}(Z_1^\delta Z_2^{1-(\delta+\gamma)})^\frac{1-\lambda}{\lambda} \tag{12}
\]
Carrying out the same steps for the \(X_2\) industry gives the profit expression
\[
\pi_{X_2} = P_{X_2} X_2 - \int_0^{N_2} P_{G_{2j}} G_{2j} dj - \int_0^{N_2} w_{l_{2j}} dj
\]
and the solutions
\[
G_{2j} = \left[\frac{\lambda P_{X_2}}{P_{G_{2j}}}\right]^{\frac{1}{1-\lambda}} Z_2^\delta Z_1^{1-(\delta+\gamma)} l_{1j} \tag{13}
\]
\[
l_{2j} = (P_{X_2} \frac{1-\lambda}{w})^{\frac{1}{\lambda}} G_{2j}(Z_2^\delta Z_1^{1-(\delta+\gamma)})^\frac{1-\lambda}{\lambda} \tag{14}
\]
Substituting the inverse demands for \(X_1\) and \(X_2\), given by 7 and 8, into 11 and 13 gives the demand functions for the \(G_{ij}\):
\[
G_{1j} = \left[\frac{\lambda e(X_1/X_2)^{r-1}}{P_{G_{1j}}}\right]^{\frac{1}{1-\lambda}} Z_1^\delta Z_2^{1-(\delta+\gamma)} l_{1j} \\
G_{2j} = \left[\frac{\lambda(1-\epsilon)(X_1/X_2)^{r'}}{P_{G_{2j}}}\right]^{\frac{1}{1-\lambda}} Z_2^\delta Z_1^{1-(\delta+\gamma)} l_{2j}
\]
We next derive the allocation of labor between the two processed good industries. Profits of processed goods firms are zero, so
\[
P_{X_1} X_1 = \int_0^{N_1} (P_{G_{1j}} G_{1j} + w_{1j}) dj \tag{15}
\]
\[
P_{X_2} X_2 = \int_0^{N_2} (P_{G_{2j}} G_{2j} + w_{l_{2j}}) dj
\]
Competition and Cobb-Douglas production in the processed good sector imply that the processed good producers pay \(\lambda P_{X_i} X_i\), \(i = 1, 2\) to the intermediate good producers, so \(\lambda P_{X_1} X_1 = \int_0^{N_1} G_{1j} P_{G_{1j}} dj\) and \(\lambda P_{X_2} X_2 = \int_0^{N_2} G_{2j} P_{G_{2j}} dj\). They pay \((1-\lambda)P_{X_i} X_i\) to the workers, so \((1-\lambda)P_{X_1} X_1 = \int_0^{N_1} w_{1j} dj\) and
(1 – λ)P_{X_1}X_2 = \int_{0}^{N_2} w l_{2j} dj. Similarly, competition and Cobb-Douglas production in the final good sector imply that P_{X_1}X_1 = \epsilon Y and P_{X_2}X_2 = (1 – \lambda)Y. We therefore have

\begin{align*}
\lambda \epsilon Y &= \int_{0}^{N_1} G_{1j} P_{G_{1j}} \lambda dj \\
\lambda (1 – \epsilon)Y &= \int_{0}^{N_2} G_{2j} P_{G_{2j}} \lambda dj \\
(1 – \lambda) \epsilon Y &= \int_{0}^{N_1} w l_{1j} dj \\
(1 – \lambda)(1 – \epsilon)Y &= \int_{0}^{N_2} w l_{2j} dj
\end{align*}

Define the total amount of labor in industries X_1 and X_2 as \( L_1 \equiv \int_{0}^{N_1} l_{1j} dj \) and \( L_2 \equiv \int_{0}^{N_2} l_{2j} dj \). Then we get the labor allocation from the ratio of the total labor compensation paid in the two industries:

\[
\frac{(1 – \lambda) \epsilon Y}{(1 – \lambda)(1 – \epsilon)Y} = \frac{\int_{0}^{N_1} w l_{1j} dj}{\int_{0}^{N_2} w l_{2j} dj} = \frac{\int_{0}^{N_1} l_{1j} dj}{\int_{0}^{N_2} l_{2j} dj} = \frac{L_1}{L_2} = \frac{\epsilon}{1 – \epsilon}
\] (16)

We can also get the above result by setting \( MPl_{1j} = MPl_{2j} \equiv w \). That derivation is omitted here.

### 1.3 Derivation of \( \frac{X_1}{X_2} \)

The ratio \( X_1/X_2 \) appears in 9 and elsewhere, so we obtain an expression for it.

\[
X_1 = \int_{0}^{N_1} G_{1j} \left( Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \right)^{1-\lambda} dj
\]

\[
\begin{align*}
&= \int_{0}^{N_1} \left( \frac{\lambda \epsilon (X_2)^{\epsilon-1}}{P_{G_{1j}}} \right)^{\lambda} \left( Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \right)^{\lambda} \left( Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \right)^{1-\lambda} dj \\
&= \int_{0}^{N_1} \left( \frac{\lambda \epsilon (X_2)^{\epsilon-1}}{P_{G_{1j}}} \right)^{\lambda} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} dj
\end{align*}
\] (17)

Similarly, we get

\[
X_2 = \int_{0}^{N_2} \left[ \frac{\lambda (1 – \epsilon)(X_2)^{\epsilon}}{P_{G_{2j}}} \right]^\lambda \left( Z_{2j}^{\delta} Z_2^{\gamma} Z_1^{1-(\delta+\gamma)} l_{2j} \right) dj
\] (18)

Taking the ratio of 17 to 18, we get

\[
\frac{X_1}{X_2} = \left( \frac{\epsilon}{1 – \epsilon} \right)^{\lambda} \left( \frac{X_1}{X_2} \right)^{-(\delta+\gamma)-1+\gamma} \left( \frac{Z_1}{Z_2} \right)^{(\delta+\gamma)-1+\gamma} \left( \frac{1}{l_{1j} P_{G_{1j}}} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{1-\lambda} \left( \frac{1}{l_{2j} P_{G_{2j}}} \right) \left( \frac{Z_1}{Z_2} \right)^{\lambda}
\]

\[
\Rightarrow \left( \frac{X_1}{X_2} \right)^{\frac{\lambda}{\lambda+1}} = \left( \frac{\epsilon}{1 – \epsilon} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{\delta+\gamma-1} \left( \frac{1}{l_{1j} P_{G_{1j}}} \right)^{\lambda} \left( \frac{1}{l_{2j} P_{G_{2j}}} \right) \left( \frac{Z_1}{Z_2} \right)^{\lambda}
\]

\[
\Rightarrow \frac{X_1}{X_2} = \left( \frac{\epsilon}{1 – \epsilon} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{(\delta+\gamma-1)(1-\lambda)} \left( \frac{1}{l_{1j} P_{G_{1j}}} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{1-\lambda}
\] (19)

### 2 Intermediate Goods Sector

The intermediate goods sector is monopolistically competitive, and each firm has the power to set its own price. The two industries in the sector are similar in structure, but notational complexities make it expedient to discuss the industries separately.
2.1 Incumbents in Industry 1

The fixed operating cost is the same for all firms in the industry, \( \theta_1 Z_1^2 / Z_2 \), where \( Z_1 \) and \( Z_2 \) are the average quality levels in the two industries and are taken as given by individual firms. The unit cost for industry 1 is \( A_1 \), i.e. 1 unit of intermediate good requires \( A_1 \) units of final good. Firm profit is

\[
F_{ij} = G_{ij} \left( P_{G_{ij}} - A_1 \right) - \theta_1 Z_1^2 / Z_2 \tag{20}
\]

Given the demand function, each firm sets its own price and devotes R&D resource to maximize the present value of profit. The current value of Hamiltonian is

\[
CV H_{ij} = G_{ij} (P_{G_{ij}} - A_1) - \theta_1 Z_1^2 / Z_2 - R_{ij} + q_{ij} (\alpha_1 I_{ij}) \tag{21}
\]

s.t. \( G_{ij} = [\lambda e(\lambda \gamma)^{-1} / P_{G_{ij}}]^{\lambda} Z_1^\delta Z_1^\delta Z_1^\delta (\delta + \gamma) l_{ij} \) and \( \dot{Z}_{ij} = \alpha_1 I_{ij} \).

Define \( r_1 \) as the return to R&D in industry 1 and \( r_2 \) as the return in R&D in industry 2. Investment \( I_1 \) is determined as follows:

\[
\begin{cases}
I_{ij} = 0 & \text{if } r_1 < r_2 \\
I_{ij} = R_{ij} & \text{if } r_1 = r_2 \\
I_{ij} = R_{ij} + \frac{N_2}{N_1} R_{2j} & \text{if } r_1 > r_2
\end{cases}
\]

Later we will see that the numbers of firms in each industry \( N_1 \) and \( N_2 \) always jump to make \( r_1 = r_2 \) and thus \( I_{ij} = R_{ij} \).

The necessary conditions are:

1. \( \dot{Z}_{ij} = \alpha_1 I_{ij} \)
2. \( \partial CV H_{ij} / \partial P_{G_{ij}} = 0 \)
3. \( \partial CV H_{ij} / \partial R_{ij} = -1 + q_{ij} \alpha_1 \)
4. \( q_{ij} = r_1 q_{ij} - \partial CV H_{ij} / \partial Z_{ij} \)
5. \( Z_{ij,t=0} \) is given
6. \( \lim_{t \to \infty} e^{-\int_0^t r(s) ds} q_{ij} (t) Z_{ij} (t) = 0 \)

Condition (2) provides the solution for the price \( P_{G_{ij}} \) of intermediate good \( G_{ij} \). Recall that \( G_{ij} = [\lambda P_{X_i} / P_{G_{ij}}]^{\lambda} Z_1^\delta Z_1^\delta Z_1^\delta (\delta + \gamma) l_{ij} \). Then

\[
0 = \frac{\partial CV H_{ij}}{\partial P_{G_{ij}}} \Rightarrow 0 = \frac{\partial (P_{G_{ij}} - A_1) G_{ij}}{\partial P_{G_{ij}}} \quad \text{because no other parts of } CV H_{ij} \text{ include } P_{G_{ij}}
\]

\[
= \frac{\partial \left[ \left( P_{G_{ij}}^{1 - \frac{1}{\lambda}} - A_1 P_{G_{ij}}^{\frac{1}{\lambda}} \right) (\lambda P_{X_i})^{\frac{1}{\lambda}} Z_1^\delta Z_1^\delta Z_1^\delta (\delta + \gamma) l_{ij} \right]}{\partial P_{G_{ij}}}
\]

\[
= -\frac{\lambda}{1 - \lambda} P_{G_{ij}}^{\frac{1}{\lambda}} + \frac{1}{1 - \lambda} A_1 \frac{1}{\lambda} P_{G_{ij}}^{\frac{1}{\lambda}} - 1
\]

\[
\Rightarrow 0 = -\frac{\lambda}{1 - \lambda} P_{G_{ij}}^{\frac{1}{\lambda}} + \frac{1}{1 - \lambda} A_1 \frac{1}{\lambda} P_{G_{ij}}^{\frac{1}{\lambda}} - 1
\]

\[
\Rightarrow P_{G_{ij}} = \frac{A_1}{\lambda}
\]

\[
\equiv P_{G_1}
\]

(22)
Each firm in industry 1 faces the same unit cost, $A_1$ units of $Y$, so the prices of all intermediate-good firms in industry 1 are the same monopoly markup over the same unit cost.

Condition (3) gives the rule for setting the value of R&D spending. Recall from above that we will show later that the rates of return to R&D in the two intermediate-good industries, $r_1$ and $r_2$, always are equal consequently R&D investment $I_{1j}$ by the firm producing $G_{1j}$ always equals that firm’s retained earnings $R_{1j}$. Then we obtain from condition (3)

$$\frac{\partial CVH_{1j}}{\partial R_{1j}} = -1 + q_{1j} \alpha_1$$

$$= 0$$

$$\implies \begin{cases} R_{1j} = \infty & \text{if } 1/\alpha_1 > q_{1j} \\ R_{1j} > 0 & \text{if } 1/\alpha_1 = q_{1j} \\ R_{1j} = 0 & \text{if } 1/\alpha_1 < q_{1j} \end{cases}$$

The interior solution, $1/\alpha_1 = q_{1j}$, is the same for all $j$, implying that in the interior all firms in industry 1 choose the same level of R&D, which we denote $R_1$.

Rearranging condition (4) allows us to express the rate of return to R&D as the profit flow plus any capital gain:

$$r_1 = \frac{\partial CVH_{1j}}{\partial Z_{1j}} + \frac{q_{1j}}{q_{1j}}$$
$$= \frac{\partial F_{1j}}{\partial Z_{1j}} + \frac{q_{1j}}{q_{1j}}$$

(23)

Condition (6) is the transversality condition.

Substituting the demand function 11 into the cash flow 20 gives

$$F_{1j} = A_1 \left( \frac{1}{\lambda} - 1 \right) \left( \frac{\lambda P_{X_1}}{P_{G_1}} \right) \frac{1}{r_1} (Z_{1j}^d Z_1^\gamma Z_2^{1-(\delta+\gamma)} l_{1j}) - \theta_1 \frac{Z_1^2}{Z_2}$$
$$= A_1 \left( \frac{1}{\lambda} - 1 \right) \left( \frac{\lambda P_{X_1}}{A_1/\lambda} \right) \frac{1}{r_1} (Z_{1j}^d Z_1^\gamma Z_2^{1-(\delta+\gamma)} l_{1j}) - \theta_1 \frac{Z_1^2}{Z_2}$$

(24)

So

$$\frac{\partial F_{1j}}{\partial Z_{1j}} = \delta A_1 \frac{1}{\lambda} \left( \frac{\lambda P_{X_1}}{A_1/\lambda} \right) \frac{1}{r_1} Z_{1j}^{\delta-1} Z_1^\gamma Z_2^{1-(\delta+\gamma)} l_{1j}$$

(25)

Using 23 and 25, we can express the return in R&D in industry 1 as

$$r_{1j} = \frac{\partial F_{1j}}{\partial Z_{1j}} \frac{1}{q_{1j}} + \frac{q_{1j}}{q_{1j}}$$
$$= \frac{\partial F_{1j}}{\partial Z_{1j}} \frac{1}{q_{1j}} + 0 \text{ because } q_{1j} = 1/\alpha_1, \text{ a constant}$$
$$= \frac{\partial F_{1j}}{\partial Z_{1j}} \frac{1}{\alpha_1}$$
$$= \alpha_1 \delta A_1 \frac{1}{\lambda} \left( \frac{\lambda P_{X_1}}{A_1/\lambda} \right) \frac{1}{r_1} Z_{1j}^{\delta-1} Z_1^\gamma Z_2^{1-(\delta+\gamma)} l_{1j}$$

(26)

We now prove symmetry among firms in the same industry. Plug the demand for $G_{1j}$ 12 into the demand for labor $l_{1j}$ 11 to obtain

$$w = (1 - \lambda) P_{X_1} \left( \frac{\lambda P_{X_1}}{P_{G_{1j}}} \right) \frac{1}{r_1} Z_{1j}^d Z_1^\gamma Z_2^{1-(\delta+\gamma)}$$
All firms in industry 1 face the same spillovers, $Z_1$ and $Z_2$, and the same price of processed good, $P_{X_1}$. They also set the same price $P_{G_{1k}}$ according to 22. If $Z_{1j} > Z_{1k}$, then all workers would be allocated to $G_{1j}$ and get paid a higher wage, leaving no demand for $G_{1k}$. The firm producing $G_{1j}$ would take over the whole industry 1 as a monopoly, and its profit would exceed its fixed operating cost, generating an excess monopoly profit.\(^1\) The entry cost is zero, however, so the monopoly profit immediately would induce new firms to enter industry 1 with new varieties carrying the average level of quality $Z_1$. The incumbent here is a monopoly, so the average quality level would be the monopoly’s quality. Entry would continue to the point where profits had been driven down to fixed operating cost, resulting in a new monopolistically competitive equilibrium in which all firms would have the same quality and thus would divide industry 1 equally. In this way, the assumptions that entry is free and that new entrants have the average quality of existing varieties ensure the symmetry among firms in the industry.

Symmetry implies that $Z_{1j} = Z_1$ for all $j$, so 34 becomes

$$
\frac{X_1}{X_2} = \frac{1}{1-\epsilon} (\frac{Z_1}{Z_2})^{(\delta + 2\gamma - 1)(1-\lambda)} (\frac{P_{G_{1j}}}{P_{G_2}})^{-\lambda} \left( \frac{\int_0^{N_j} Z_{1j}^{\delta + 1} dj}{\int_0^{N_j} Z_{2j}^{\delta + 1} dj} \right)^{1-\lambda}
$$

Substitute this expression into the inverse demand function 7 and then substitute the result into 26 to get

$$
P_{X_1} = \epsilon X_1^{\epsilon - 1} X_2^{1 - \epsilon}
= \epsilon (\frac{\epsilon}{1+ \epsilon}) ((\epsilon - 1))(\frac{Z_1}{Z_2})^{(\delta + 2\gamma - 1)(1-\lambda)(\epsilon - 1)} (\frac{P_{G_{1j}}}{P_{G_2}})^{-\lambda(\epsilon - 1)}
$$

We thus have

$$
r_1 = \alpha_1 \delta A_1 \frac{1 - \lambda - \frac{\lambda P_{X_1}}{A_1}}{\lambda} \frac{1}{\lambda^2} Z_{1j}^{\delta + \gamma - 1} Z_2^{1 - (\delta + \gamma)} l_{1j}
= \delta \alpha_1 A_1 \frac{1 - \lambda - \frac{\lambda P_{X_1}}{A_1}}{\lambda} \frac{1}{\lambda^2} (\frac{\epsilon}{1-\epsilon}) \left( \frac{\frac{P_{G_{1j}}}{P_{G_2}}}{\frac{Z_2}{Z_1}} \right)^{\frac{\lambda(\epsilon - 1)}{\lambda}} l_{1j}
$$

where $\Gamma = \delta + \gamma + [2(\delta + \gamma) - 1](\epsilon - 1) \in (0, 1)$. The restriction that $\Gamma \in (0, 1)$ can be seen in two steps. First, $\Gamma$ is positive:

$$
\delta + \gamma + [2(\delta + \gamma) - 1](\epsilon - 1) = \delta + \gamma + (1 - 2(\delta + \gamma))(1 - \epsilon)
> 0
$$
given that $\gamma > 0$, $\delta > 0$, and $0 < \delta + \gamma < 1$. Second, $\Gamma < 1$:

$$
\delta + \gamma + [2(\delta + \gamma) - 1](\epsilon - 1) = 2(\delta + \gamma)\epsilon - (\delta + \gamma) - \epsilon + 1
= 1 - (\delta + \gamma)(1 - \epsilon) - \epsilon[1 - (\delta + \gamma)]
< 1
$$
given that $\delta + \gamma < 1$ and $\epsilon \in (0, 1)$.

\(^1\)Note that the monopoly firm still sets price as 22, even though it is facing a larger market size. That is because the price elasticity of the demand is constant.
In Section (2.4) we will see that zero entry cost induces entry if incipient profits arise. In equilibrium, distributed profit \( \pi_{ij} = F_{ij} - R_{ij} \) is zero, i.e.,

\[
0 = \pi_{ij} = F_{ij} - R_{ij} = G_{ij} \left( P_{G_{ij}} - A_1 \right) - \theta_1 Z_2^2 - R_{ij}
\]

\[
= A_1 \frac{1 - \lambda}{\lambda} \left( \frac{\lambda}{A_1} \right)^{t-x} \left( \frac{\epsilon}{1 - \epsilon} \right)^{\frac{1}{1 - \lambda}} \left( \frac{P_{G_{ij}}}{P_{G_{2}}} \right)^{\frac{-\lambda(\alpha - 1)}{1 - \lambda}} Z_1^1 \left( Z_2^2 \right)^{-1 - \gamma} l_{ij} - \theta_1 Z_2^2 - R_{ij}
\]

We have \( L_1 \equiv \int_0^{N_1} l_{ij} \). From 16 and \( L_1 + L_2 = L \), we know that the total market size for industry 1, \( L_1 \), is fixed. Thus entry makes \( l_{ij} \) equal among firms in the industry. We also have \( l_1 = L_1/N_1 \).

We have obtained the results that all firms choose the same \( R_{ij} = R_1 \) and the same output prices \( P_{G_{ij}} = A_1/\lambda \equiv P_{G_{1}} \). Thus all firms choose the same levels of the two things they control, output price and R&D spending. Because all firms in the industry make the same choices, we henceforth drop the firm subscript.

### 2.2 Incumbents in Industry 2

Following the similar steps we get the results in industry 2.

\[
CVH_2 = G_2(P_{G_2} - A_2) - \theta_2 Z_2^2 - R_2 + q_2(\alpha_2 I_2)
\]

s.t. \( G_2 = \left[ \lambda P_{X_2}/P_{G_2} \right] \frac{\lambda}{\lambda - 1} \left( Z_2^2 \right)^{1-\gamma} (Z_2^2)_{l_2} Z_1^{1-(\delta + \gamma)} l_2/N_2 \) and \( \dot{Z}_2 = \alpha_2 R_2 \). As before, \( r_1 = r_2 \), so \( I_2 = R_2 \). The first-order condition \( \partial CVH_2 \)/\( \partial P_{G_2} = 0 \) implies

\[
P_{G_2} = \frac{A_2}{\lambda}
\]

The first-order condition \( \partial CVH_2 / \partial R_2 = -1 + q_2 \alpha_2 \) implies

\[
\begin{align*}
R_2 &= \infty \quad \text{if } 1/\alpha_2 > q_2 \\
R_2 &= 0 \quad \text{if } 1/\alpha_2 = q_2 \\
R_2 &= 0 \quad \text{if } 1/\alpha_2 < q_2
\end{align*}
\]

The transversality condition is \( \lim_{t \to -\infty} e^{-\int_t^{\infty} r(s) ds} q_{2j}(t) Z_{2j}(t) = 0 \). Profit is

\[
F_2 = G_2(P_{G_2} - A_2) - \theta_2 Z_2^2 - R_2
\]

\[
= A_2 \left( \frac{1}{\lambda} - 1 \right) \left( \frac{\lambda P_{X_2}}{P_{G_2}} \right)^{t-x} \left( Z_2^2 \right)^{1-\gamma} (Z_2^2)_{l_2} Z_1^{1-(\delta + \gamma)} l_2 - \theta_2 Z_2^2
\]

\[
= A_2 \left( \frac{1 - \lambda}{\lambda} \right) \left( \frac{\lambda P_{X_2}}{A_2/\lambda} \right)^{t-x} (Z_2^2)_{l_2} Z_2^2 Z_1^{1-(\delta + \gamma)} l_2 - \theta_2 Z_2^2
\]

where

\[
P_{X_2} = (1 - \epsilon) \left( \frac{X_1}{X_2} \right)^{\delta}
\]

\[
= (1 - \epsilon) \left( \frac{\epsilon}{1 - \epsilon} \right)^{\gamma} \left( \frac{Z_1}{Z_2} \right)^{(2\delta + 2\gamma - 1)(1 - \lambda) + \lambda \epsilon} \left( \frac{P_{G_{ij}}}{P_{G_2}} \right)^{\lambda \epsilon}
\]

9
We then obtain
\[
\frac{\partial F_2}{\partial Z_{2j}} = \delta A_2 \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda P_{X_2}}{A_2/\lambda} \right]^{1/\gamma} (Z_2^{\delta + \gamma - 1} Z_1^{1 - (\delta + \gamma)}) l_2
\]
(32)
\[
= \delta A_2 \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_2} \right]^{1/\gamma} \left( \frac{\epsilon}{1 - \epsilon} \right)^{1/\gamma} (Z_2)^{\delta + \gamma - 1} (Z_1)^{1 - (\delta + \gamma)} (P_{G_1} / P_{G_2})^{1/\gamma} Z_2^{\delta - 1} Z_1^{1 - \delta} l_2
\]
and
\[
r_2 = \frac{\partial F_2}{\partial Z_2} \frac{1}{q_2} = \frac{\partial F_2}{\partial Z_2} \frac{1}{q_2} = 0 = \frac{\partial F_2}{\partial Z_2} \alpha_2
\]
(33)
\[
= \delta \alpha_2 A_2 \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_2} \right]^{1/\gamma} \left( \frac{\epsilon}{1 - \epsilon} \right)^{1/\gamma} (P_{G_1} / P_{G_2})^{1/\gamma} (Z_1 / Z_2)^{1 - \delta} l_2
\]

Because \( 1 > \Gamma > 0 \), there are diminishing returns to \( Z_2 \).

We obtain all the same symmetry results as before. Thus firms within each industry are symmetric. However, in general \( P_{G_1} = A_1 / \lambda \neq A_2 / \lambda = P_{G_2} \) and also \( I_1 = R_1 \neq R_2 = I_2 \) because in general \( 1/\alpha_1 \neq 1/\alpha_2 \). Thus firms differ across the two industries, and the model is one of partially asymmetric equilibrium.

2.3 The Ratio \( X_1 / X_2 \) Again

Using 19 and 16, we can write \( \frac{X_1}{X_2} \) as
\[
\frac{X_1}{X_2} = \left( \frac{\epsilon}{1 - \epsilon} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{(\delta + 2\gamma - 1)(1 - \lambda)} \left( \frac{\int_0^{N_1} l_{1j} P_{G_1}^{-\lambda} Z_{1j}^{\delta} dj}{\int_0^{N_2} l_{2j} P_{G_2}^{-\lambda} Z_{2j}^{\delta} dj} \right)^{1 - \lambda}
\]
(34)
\[
= \left( \frac{\epsilon}{1 - \epsilon} \right)^{\lambda} \left( \frac{Z_1}{Z_2} \right)^{(\delta + 2\gamma - 1)(1 - \lambda)} \left( \frac{P_{G_1}}{P_{G_2}} \right)^{1 - \lambda} \left( \frac{\int_0^{N_1} l_{1j} Z_{1j}^{\delta} dj}{\int_0^{N_2} l_{2j} Z_{2j}^{\delta} dj} \right)^{1 - \lambda}
\]

2.4 Zero Entry Cost, Zero Profit, and the Growth Rates of \( Z_1 \) and \( Z_2 \)

Zero entry cost implies that the zero profit condition holds at all times (see the main text for more explanation). Distributed profit is \( \Pi_i = F_i - R_i \), so the zero profit condition requires that \( R_i = F_i \). Then using \( Z_1 = \alpha_1 R_1 \) and \( Z_2 = \alpha_2 R_2 \) with the zero profit condition, 24, and 31, we get the growth rates of \( Z_1 \) and \( Z_2 \):
\[
g_1 = \frac{\dot{Z}_1}{Z_1} = \frac{\alpha_1 R_1}{Z_1} = \frac{\alpha_1 F_1}{Z_1}
\]
(35)
\[
= \alpha_1 \left\{ A_1 \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_1} \right]^{1/\gamma} \left( \frac{\epsilon}{1 - \epsilon} \right)^{1/\gamma} (P_{G_1} / P_{G_2})^{1/\gamma} \left( \frac{L_1}{Z_1} \right)^{\beta - 1} - \theta_1 \frac{Z_1}{Z_2} \right\}
\]
\[
g_2 = \frac{\dot{Z}_2}{Z_2} = \frac{\alpha_2 R_2}{Z_2} = \frac{\alpha_2 F_2}{Z_2}
\]
(36)
\[
= \alpha_2 \left\{ A_2 \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_2} \right]^{1/\gamma} \left( \frac{\epsilon}{1 - \epsilon} \right)^{1/\gamma} (P_{G_1} / P_{G_2})^{1/\gamma} \right\}
\]
3 General Equilibrium

3.1 Market Clearing Conditions

There are three markets that must clear simultaneously: labor, goods, and credit (or asset).

3.1.1 Labor Market

Labor supply is
\[ L = L_1 + L_2 = \int_0^{N_1} l_{1j}dj + \int_0^{N_2} l_{2j}dj \]  
(37)

Labor demand is given by equations 12 and 14, and from 16 we know
\[ \frac{L_1}{L_2} = \frac{\epsilon}{1 - \epsilon} \]

so
\[ L_1 = \epsilon L \]  
(38)
\[ L_2 = (1 - \epsilon)L \]  
(39)

3.1.2 Goods Market

The resource constraint is
\[ Y = \int_0^{N_1} A_{1j}G_{1j}dj + \int_0^{N_1} R_{1j}dj + \int_0^{N_1} \phi_{1j}dj + \int_0^{N_2} A_{2j}G_{2j}dj \]
\[ + \int_0^{N_2} R_{2j}dj + \int_0^{N_2} \phi_{2j}dj + C \]  
(40)
\[ + \int_0^{N_2} R_{2j}dj + \int_0^{N_2} \phi_{2j}dj + C \]  
(41)

where \( \phi_{1j} = \theta_1Z_1^2/Z_2 \), \( \phi_{2j} = \theta_2Z_2^2/Z_1 \).

The factor payments are the following:

1. Final good producers pay \( P_{X_1}X_1 \) to the firms in the \( X_1 \) sector. Competition and Cobb-Douglas production guarantee that \( P_{X_1}X_1 = \epsilon 1 \cdot Y \).
2. Similarly, final good producers pay \( P_{X_2}X_2 = (1 - \epsilon)1 \cdot Y \) to the firms in the \( X_2 \) sector.
3. The firms in the \( X_1 \) sector pay \( \int_0^{N_1} P_{G_{1j}}G_{1j}dj = \lambda P_{X_1}X_1 \) to the firms in the \( G_1 \) sector.
4. The firms in the \( X_1 \) sector pay \( \int_0^{N_1} w_{1j}dj = (1 - \lambda)P_{X_1}X_1 \) to labor.
5. The firms in the \( X_2 \) sector pay \( \int_0^{N_2} P_{G_{2j}}G_{2j}dj = \lambda P_{X_2}X_2 \) to the firms in the \( G_2 \) sector.
6. The firms in the \( X_2 \) sector pay \( \int_0^{N_2} w_{2j}dj = (1 - \lambda)P_{X_2}X_2 \) to labor.

Summing these, we can write
\[ 1 \cdot Y = P_{X_1}X_1 + P_{X_2}X_2 \]
\[ = \int_0^{N_1} P_{G_{1j}}G_{1j}dj + \int_0^{N_1} w_{1j}dj + \int_0^{N_2} P_{G_{2j}}G_{2j}dj + \int_0^{N_2} w_{2j}dj \]  
(42)
The zero profit condition requires \( P_{G_1} G_1 = A_1 G_1 + R_1 + \phi_1 j \) and \( P_{G_2} G_2 = A_2 G_2 + R_2 + \phi_2 j \). So (42) can be written as

\[
Y = \int_0^{N_1} A_1 G_1 dj + \int_0^{N_1} R_1 dj + \int_0^{N_1} \phi_1 j dj + \int_0^{N_2} A_1 G_2 dj + \int_0^{N_2} R_2 dj + \int_0^{N_2} \phi_2 j dj + wL
\]  

Comparing (40) and (43), we get

\[
C = wL = (1 - \lambda)(P_{X_1} X_1 + P_{X_2} X_2) = (1 - \lambda)Y
\]

Dividing by \( Y \) gives

\[
\frac{wL}{Y} = \frac{C}{Y} = 1 - \lambda
\]

which implies

\[
\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{w}}{w}
\]

because \( L \) and \( 1 - \lambda \) are constant.

### 3.1.3 Credit Market (No Arbitrage Condition)

No arbitrage requires that all rates of return are equal:

\[
r_1 = r_2 = r
\]

so the right sides of (26) and (33) must be equal. If one return is higher, then all resources go to accumulating that quality. The numbers of firms \( N_1 \) and \( N_2 \) in each intermediate good industry jump up or down, which lowers or raises firm size \( l_1 \) and \( l_2 \). From 26 and 31, changes in firm size lower or raise the rates of return \( r_1 \) and \( r_2 \). Those changes continue until the arbitrage opportunity has been eliminated. See section 4 below for more discussion.

### 3.2 Derivations of the Growth Rates \( \frac{G_1}{G_1}, \frac{G_2}{G_2}, \frac{X_1}{X_1}, \frac{X_2}{X_2}, \frac{Y}{Y}, \frac{\dot{w}}{w}, \) and \( \frac{\dot{C}}{C} \)

Substitute the inverse demand for \( X_1 \) and the expression (27) for \( X_1 / X_2 \) into demand for \( G_1 \), (11) and rearrange terms to obtain

\[
G_1 = \frac{\lambda e}{P_{G_1}} \left( \frac{e}{1 - e} \right)^{1 - \lambda} \left( \frac{P_{G_1}}{P_{G_2}} \right)^{-\frac{\lambda e - 1}{e - 1}} \left( \frac{Z_1}{Z_2} \right)^{-1 - 2(\delta + \gamma)/(1 - e)} (1 - 2(\delta + \gamma))/(1 - e) \left( Z_1^e Z_2^{1 - (\delta + \gamma)} \frac{L_1}{N_1} \right)
\]

where \( \xi_1 = (\lambda e/P_{G_1}) \left[ \frac{1}{e} - (1/1 - e) \right] / (1 - e) \) \( (e - 1)/(1 - e) \) \( (P_{G_1}/P_{G_2})^{-\lambda e - 1}/(1 - e) \). The prices \( P_{G_1} \) and \( P_{G_2} \) are constant and equal to the unit cost times the monopolistic markup. The number of firms \( N_1 \) is a jumping variable. Labor in the first intermediate goods industry is the constant \( L_1 = eL \). The only variables affecting the growth rate of \( G_1 \) are \( Z_1 \) and \( Z_2 \). For given \( L_1 / N_1 \),

\[
\frac{\dot{G}_1}{G_1} = \Gamma \left( \frac{\dot{Z}_1}{Z_1} + (1 - \Gamma) \frac{\dot{Z}_2}{Z_2} \right)
\]

where as before \( \Gamma \equiv [1 - 2(\delta + \gamma)] / (1 - e) + \delta = 1 - e - (\delta + \gamma) + 2(\delta + \gamma)e \). By a similar derivation for \( G_2 \), we obtain

\[
G_2 = \xi_2 Z_1 Z_2^{1 - \epsilon - (\delta + \gamma) + 2(\delta + \gamma) e} \left( Z_2 e^{(\delta + \gamma) - 2e(\delta + \gamma)} \frac{L_2}{N_2} \right)
\]
where \( \xi_2 = (\lambda(1-\epsilon)/P_G^2)^{1/(1-\lambda)}(\epsilon/(1-\epsilon))^{(1-\lambda)/(1-\lambda)}(P_G/P_G^2)^{-\lambda\epsilon/(1-\lambda)} \). The growth rate of \( G_2 \) is

\[
\frac{\dot{G}_2}{G_2} = \Gamma \frac{\dot{Z}_1}{Z_1} + (1-\Gamma) \frac{\dot{Z}_2}{Z_2}
\]

To obtain the growth rate of \( X_1 \), we must write the \( X_1 \) in terms of the quality levels \( Z_1 \) and \( Z_2 \). Plug the demand function for \( G_1 \) 11 into 10 to get 17. Combine 27, 38 and 17 and rearrange terms to obtain

\[
X_1 = \lambda(1-\epsilon)(\frac{X_1}{X_2})^{\epsilon-1} \rightarrow \frac{\lambda(1-\epsilon)}{X_2} Z_1^{\delta+\gamma} Z_1^{-1-(\delta+\gamma)} L_1
\]

\[
= \lambda(1-\epsilon) \rightarrow \frac{X_1}{X_2} \frac{\lambda(1-\epsilon)}{X_2} Z_1^{\delta+\gamma} Z_1^{-1-(\delta+\gamma)} L_1
\]

\[
= (\lambda(1-\epsilon)) \frac{X_1}{X_2} Z_1^{\delta+\gamma} Z_1^{-1-(\delta+\gamma)} L_1
\]

\[
= \zeta_1 Z_1^{\delta+\gamma} Z_1^{-1-(\delta+\gamma)} L_1
\]

where \( \zeta_1 = (\lambda(1-\epsilon))^{\lambda/(1-\lambda)}(\epsilon/(1-\epsilon))^{\lambda/(1-\lambda)}(P_G/P_G^2)^{\lambda\epsilon/(1-\lambda)} \). We easily obtain the growth rate of \( X_1 \) from 48:

\[
\frac{\dot{X}_1}{X_1} = \vartheta_1 \frac{\dot{Z}_1}{Z_1} + (1-\vartheta_1) \frac{\dot{Z}_2}{Z_2}
\]

where \( \vartheta_1 = \delta + \gamma + [1-2(\delta+\gamma)](1-\epsilon) \).

The growth rate of \( X_2 \) is obtained by similar steps. Combine 34, 39 and 18 and rearrange terms to get

\[
X_2 = N_2 \frac{\lambda(1-\epsilon)(\frac{X_1}{X_2})^{\epsilon}}{P_G^2} \rightarrow \frac{\lambda(1-\epsilon)}{P_G^2} Z_2^{\delta+\gamma} Z_2^{-1-(\delta+\gamma)} L_2
\]

\[
= \frac{\lambda(1-\epsilon)}{P_G^2} Z_2^{\delta+\gamma} Z_2^{-1-(\delta+\gamma)} L_2
\]

\[
= [\lambda(1-\epsilon)]^{\lambda/(1-\lambda)}(\epsilon/(1-\epsilon))^{\lambda/(1-\lambda)} P_G^{-\lambda/(1-\lambda)} (P_G/P_G^2) \frac{\lambda\epsilon}{(1-\epsilon)} L_2
\]

where \( \zeta_2 = [\lambda(1-\epsilon)]^{\lambda/(1-\lambda)}(\epsilon/(1-\epsilon))^{\lambda/(1-\lambda)}(P_G/P_G^2) \lambda\epsilon/(1-\lambda) \). The growth rate of \( X_2 \) is

\[
\frac{\dot{X}_2}{X_2} = \vartheta_2 \frac{\dot{Z}_1}{Z_1} + (1-\vartheta_2) \frac{\dot{Z}_2}{Z_2}
\]

where \( \vartheta_2 = 1 - (\delta + \gamma) - [1-2(\delta+\gamma)](1-\epsilon) \).

We turn now to the growth rate of \( Y \). From 1 we have \( Y = X_1^{1-\epsilon} X_2^{1-\epsilon} \). Combine 48, 50, and 1 to get

\[
Y = X_1^{1-\epsilon} X_2^{1-\epsilon}
\]

\[
= \zeta_1 Z_1^{\delta+\gamma} Z_2^{1-1-(\delta+\gamma)} L_1
\]

\[
= \zeta_2 Z_2^{1-1-(\delta+\gamma)} L_2
\]

\[
= \kappa Z_1^{\delta+\gamma} L_1
\]

\[
= \kappa Z_2^{\delta+\gamma} L_2
\]
where
\[ \kappa = (\kappa_1 \kappa_2^{1-\varrho})(\varrho)(1 - \varrho)^{1-\varrho} \]
\[ = \lambda^{\frac{\varrho}{1-\varrho}} (1 - \varrho) \frac{\lambda^{1-\varrho}}{1-\varrho} \epsilon^{\frac{\lambda}{1-\varrho}} P_{G_1} \frac{\lambda^{1-\varrho}}{1-\varrho} \epsilon^{(1 - \varrho)^{(1-\varrho)}} \]

The growth rate of $Y$ is
\[ \dot{Y} = \frac{\dot{Z}_1}{Z_1} + (1 - \Gamma) \frac{\dot{Z}_2}{Z_2} \]

Note also that $\Gamma = \varrho_1 + (1 - \varrho) \varrho_2$, which connects the growth rate of $Y$ to the growth rates of $X_1$ and $X_2$.

Finally, constant population means that consumption per person grows at the same rate as aggregate consumption:
\[ \frac{\dot{C}}{C} = \frac{\dot{\psi}}{\psi} = \dot{Y} = \frac{\dot{Z}_1}{Z_1} + (1 - \Gamma) \frac{\dot{Z}_2}{Z_2} \]

(53)

3.3 Euler Equation

The household maximizes lifetime utility $U(t) = \int_{t}^{\infty} \log(c) e^{-\rho t} \, dt$ s.t. $\dot{S} = rS + wL - cL$, where $c = C/L$ is consumption per capita, $S$ is asset holdings, and $r$ is the rate of return on assets. The current value Hamiltonian is
\[ H = \log(c_t) + \psi_t (rS + wL - cL) \]
where $\psi_t = e^{\rho t} \lambda_t$ is the current value costate variable. The necessary conditions are:
\[ \dot{S} = rS + wL - cL \]
\[ \dot{\psi} = \rho \psi_t - \frac{\partial H}{\partial S_t} = (\rho - r) \psi_t \]
\[ \frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \psi_t = 0 \]
\[ S_0 \text{ given} \]
\[ \lim_{t \to \infty} e^{-\rho t} \psi_t S_t = 0 \]

Use the Euler equation $1/c_t = \psi_t$ and $\dot{\psi} = (\rho - r) \psi_t$ to obtain
\[ -\frac{\dot{c}_t}{c_t} = (\rho - r) \frac{1}{c_t} \]
\[ \Rightarrow \frac{\dot{c}_t}{c_t} = (r - \rho) \]

(54)

4 Balanced Growth Path and Transition Dynamics

We now derive the dynamic behavior of the economy.
4.1 Balanced Growth Path

We begin with the balanced growth path (BGP).

We have nine unknowns: \(g_1, g_2, r_1, r_2, L_1, L_2, Z_1/Z_2, N_1,\) and \(N_2\). We determine them with the nine equations 28, 33, 35, 36, 38, 39, 46, 54, and the BGP requirement that all growth rates be constant:

\[
\frac{d \dot{c}}{dt} = \frac{d \dot{C}}{dt} = \frac{d \dot{w}}{dt} = \frac{d \dot{Y}}{dt} = \frac{d \dot{g_1}}{dt} = \frac{d \dot{g_2}}{dt} = 0 \tag{55}
\]

For the sake of notational ease, we write 28 and 33 in simplified forms:

\[
r_1 = \delta \alpha_1 A_1 \frac{1 - \lambda}{\lambda} \left( \frac{Z_1}{Z_2} \right)^{1 - \epsilon} \left( \frac{\epsilon}{1 - \epsilon} \right) \left( \frac{P_{G_1}}{P_{G_2}} \right)^{\frac{\lambda(1 - \epsilon)}{\lambda - \epsilon}} \left( \frac{Z_1}{Z_2} \right)^{-1} \frac{L_1}{N_1} \tag{56}
\]

\[
r_2 = \delta \alpha_2 A_2 \frac{1 - \lambda}{\lambda} \left( \frac{Z_1}{Z_2} \right)^{1 - \epsilon} \left( \frac{\epsilon}{1 - \epsilon} \right) \left( \frac{P_{G_1}}{P_{G_2}} \right)^{-\frac{\lambda(1 - \epsilon)}{\lambda - \epsilon}} \left( \frac{Z_1}{Z_2} \right)^{-1} \frac{L_2}{N_2} \tag{57}
\]

where the functions \(f_1\) and \(f_2\) comprise all the terms after \(\delta\) in the first lines of 56 and 57, respectively.

We use these simplified versions to write simplified versions of 35 and 36 as well:

\[
g_1 = f_1 \left( \frac{L_1}{N_1}, \frac{Z_1}{Z_2} \right) - \alpha_1 \psi_1 \left( \frac{Z_1}{Z_2} \right) \tag{58}
\]

\[
g_2 = f_2 \left( \frac{L_2}{N_2}, \frac{Z_1}{Z_2} \right) - \alpha_2 \psi_2 \left( \frac{Z_2}{Z_1} \right) \tag{59}
\]

Rearrange 58 to get

\[
f_1 = g_1 + \alpha_1 \psi_1 \left( \frac{Z_1}{Z_2} \right) \tag{60}
\]

Plug 60 into 56:

\[
r_1 = \delta [g_1 + \alpha_1 \psi_1 \left( \frac{Z_1}{Z_2} \right)] \tag{61}
\]

The no arbitrage condition 46 requires that all rates of return are equal at all times, and on the BGP all growth rates are constant. Therefore the growth rate for consumption per person, given by the Euler equation 54, requires that \(r_1\) is a constant. That then implies through 61 that the ratio \(Z_1/Z_2\) is constant, which in turn implies that \(g_1\) equals \(g_2\). From 53, we see that

\[
\frac{\dot{c}}{c} = \Gamma g_1 + (1 - \Gamma) g_2
\]

\[
= \Gamma g_1 + (1 - \Gamma) g_1
\]

\[
= g_1
\]

Substituting this result into the Euler equation 54, we obtain

\[
g_1^* = \frac{\dot{Z}_1}{Z_1} = \frac{\delta}{1 - \delta} \alpha_1 \psi_1 \left( \frac{Z_1}{Z_2} \right)^* - \frac{1}{1 - \delta} \rho \tag{62}
\]

and

\[
r_1^* = g_1^* + \rho = \frac{\delta}{1 - \delta} \alpha_1 \psi_1 \left( \frac{Z_1}{Z_2} \right)^* - \frac{1}{1 - \delta} \rho + \rho \tag{63}
\]
We proceed similarly to get the BGP values of \( g_*^2 \) and \( r_2^* \):

\[
g_*^2 = \frac{Z_2}{Z_2} = \frac{\delta}{1-\delta} \alpha_2 \theta_2 \left( \frac{Z_2}{Z_1} \right)^* - \frac{1}{1-\delta} \rho \tag{64}
\]

\[
r_2^* = g_*^2 + \rho = \frac{\delta}{1-\delta} \alpha_2 \theta_2 \left( \frac{Z_2}{Z_1} \right)^* - \frac{1}{1-\delta} \rho + \rho \tag{65}
\]

Because \( g_1 = g_2 \), we set the right sides of 62 and 64 equal to each other and solve for \( Z_1/Z_2 \):

\[
\frac{\delta}{1-\delta} \alpha_1 \theta_1 \left( \frac{Z_1}{Z_2} \right)^* - \frac{1}{1-\delta} \rho = \frac{\delta}{1-\delta} \alpha_2 \theta_2 \left( \frac{Z_2}{Z_1} \right)^* - \frac{1}{1-\delta} \rho
\]

\[
\Rightarrow \alpha_1 \theta_1 \left( \frac{Z_1}{Z_2} \right)^* = \alpha_2 \theta_2 \left( \frac{Z_2}{Z_1} \right)^*
\]

\[
\Rightarrow \alpha_1 \theta_1 \left( \frac{Z_1}{Z_2} \right)^* - \alpha_2 \theta_2 = 0 \tag{66}
\]

This quadratic has two roots

\[
\left( \frac{Z_1}{Z_2} \right)^* = \pm \sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}} \tag{67}
\]

The negative root is economically meaningless, so

\[
\left( \frac{Z_1}{Z_2} \right)^* = \sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}} > 0
\]

If we plug 67 into 62 and 63, we get the balanced growth rate:

\[
g^* = g_1^* = \frac{\delta}{1-\delta} \sqrt{\alpha_1 \theta_1 \alpha_2 \theta_2} - \frac{1}{1-\delta} \rho \tag{68}
\]

\[
= g_2^* \tag{69}
\]

with the properties \( \partial g^*/\partial (\alpha_1 \theta_1) > 0 \) and \( \partial g^*/\partial (\alpha_2 \theta_2) > 0 \).

We immediately obtain the BGP solutions \( L_1^* \) and \( L_2^* \) from 38 and 39:

\[
L_1^* = \epsilon L \tag{70}
\]

\[
L_2^* = (1-\epsilon) L \tag{71}
\]

because \( \epsilon \) and \( L \) are constant. The values \( L_1^* \) and \( L_2^* \) are the total amounts of labor employed in the \( X_1 \) and \( X_2 \) industries, respectively.

To get \( N_1 \), we begin by substituting the solutions for \( P_{G1}, P_{G2}, \) and \( (Z_1/Z_2)^* \) from 22, 30, and 67 into 28 to obtain

\[
r_1^* = \alpha_1 \delta A_1 \frac{1-\lambda}{\lambda} \sqrt{\alpha_1 \theta_1 \alpha_2 \theta_2} \left( \frac{\epsilon}{1-\epsilon} \right)^{\frac{\lambda}{1-\lambda}} \left( A_1 A_2 \right)^{\frac{\lambda-\lambda}{\lambda+\lambda}} \left( 1-\Gamma \right) \frac{L_1}{N_1} \tag{72}
\]

We then substitute 72 and 68 into 63 and rearrange terms to get

\[
L_1^* = \frac{L_1^*}{N_1^*} = \frac{\epsilon L}{N_1^*} = \frac{\delta}{1-\delta} \sqrt{\alpha_1 \theta_1 \alpha_2 \theta_2} \left( \frac{\epsilon}{1-\epsilon} \right)^{\frac{\lambda}{1-\lambda}} \left( 1-\Gamma \right) \frac{L_1}{A_1 A_2 A_2^{\frac{\lambda-\lambda}{\lambda+\lambda}}} \tag{73}
\]

\[
\Rightarrow N_1^* = \frac{\alpha_1 \delta \frac{1-\lambda}{\lambda} \left( \frac{\epsilon}{1-\epsilon} \right)^{\frac{\lambda}{1-\lambda}} \left( \frac{L_1}{A_1 A_2 A_2^{\frac{\lambda-\lambda}{\lambda+\lambda}}} \right)^{\frac{\lambda}{1-\lambda}} - \frac{\delta \rho}{1-\delta} \rho}{\frac{\alpha_1 \delta \frac{1-\lambda}{\lambda} \left( \frac{\epsilon}{1-\epsilon} \right)^{\frac{\lambda}{1-\lambda}} \left( \frac{L_1}{A_1 A_2 A_2^{\frac{\lambda-\lambda}{\lambda+\lambda}}} \right)^{\frac{\lambda}{1-\lambda}} - \frac{\delta \rho}{1-\delta} \rho}} \epsilon L \tag{74}
\]
which have the properties \( \partial l^*_1 / \partial A_1 > 0, \partial l^*_1 / \partial A_2 > 0, \partial N^*_1 / \partial A_1 < 0, \) and \( \partial N^*_1 / \partial A_2 < 0. \) Firm size \( L_1 / N_1 \) is constant on the BGP. If \( L_1 \) increases, \( N_1 \) jumps (zero entry cost) to keep the ratio constant. Consequently entry kills the scale effect at the aggregate level. We also see from 73 and 68 that the unit costs \( A_1 \) and \( A_2 \) do not enter the growth rate, though differences in unit costs do affect firm size. For example, an increase in unit cost \( A_2 \) causes an incipient decrease in profit. That has a negative effect on \( r_1 \) according to 28, and induces firms to leave instantaneously without any cost. As a result, firm size increases, which has a positive effect on \( r_1 \) according to 28. The two effects cancel, leaving \( r_1 \) unchanged. The reason that firm size \( l_1 \) is positively affected by unit cost \( A_2 \) in the other industry is that, when \( A_2 \) increases, \( P_{X_1} \) decreases according to inverse demand function for \( X_1, 7 \) and 19. Thus demand for \( G_{1j} \) decreases according to demand function of \( G_{1j}, 11. \) The result is a lower profit, inducing firms to leave.

By a similar analysis, we obtain the BGP values for \( l_2^* \) and \( N_2^* : \)

\[
l_2^* = \frac{L_2^*}{N_2^*} = \frac{(1 - \epsilon)L}{N_2} = \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_2 - \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1}} \left( \frac{\alpha_2 \alpha_2 \delta}{\alpha_1 \alpha_1 \delta} \right) \left( \frac{\alpha_2 \alpha_1 \alpha_2 \alpha_2 - \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1}}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1} \right)
\]

and

\[
N_2^* = \frac{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_2 - \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1}}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1} \left( \frac{\alpha_2 \alpha_1 \alpha_2 \alpha_2 - \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1}}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1} \right) \left( \frac{\alpha_2 \alpha_1 \alpha_2 \alpha_2 - \frac{\delta}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1}}{\alpha_2 \delta L \alpha_1 \alpha_2 \alpha_1} \right)
\]

On the BGP

\[
\frac{\dot{c}}{c} = \frac{\dot{C}}{C} = \frac{\dot{w}}{w} = \frac{\dot{Y}}{Y} = \frac{\dot{Z}_1}{Z_1} = \frac{\dot{Z}_2}{Z_2} = r - \rho
\]

4.2 Transition Dynamics

During the transition to the BGP, the growth rates of \( Z_1 \) and \( Z_2 \) generally will not be equal. Nonetheless, at all times

\[
\frac{\dot{C}}{C} = \frac{\dot{w}}{w} = \frac{\dot{Y}}{Y} = \frac{\dot{Z}_1}{Z_1} = \frac{\dot{Z}_2}{Z_2} = r - \rho
\]

from 53. From ?? and ??, we get

\[
\frac{Z_1 / Z_2}{Z_1 / Z_2} = \frac{Z_1 / Z_2}{Z_1 / Z_2} = \frac{Z_1 / Z_2}{Z_1 / Z_2} = \frac{Z_1 / Z_2}{Z_1 / Z_2}
\]

The no arbitrage condition 46 together with \( 26, 56 \) and 57 implies that

\[
f_1(L_1 / N_1, Z_1 / Z_2) = f_2(L_2 / N_2, Z_1 / Z_2)
\]

so

\[
\frac{Z_1 / Z_2}{Z_1 / Z_2} = \frac{Z_1 / Z_2}{Z_1 / Z_2} = \frac{Z_1 / Z_2}{Z_1 / Z_2}
\]

Multiply both sides by \( Z_1 / Z_2 \) to get

\[
(Z_1 / Z_2) = -\alpha_1 \theta_1 (\frac{Z_1}{Z_2})^2 + \alpha_2 \theta_2
\]
The two steady state values of $Z_1/Z_2$ are

$$\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}} > 0$$

$$-\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}} < 0$$

The steady state $\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$ is stable because $(Z_1/Z_2) \geq 0$ as $Z_1/Z_2 \leq \sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$ for all $Z_1/Z_2 > -\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$. The steady state $-\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$ is unstable because $(Z_1/Z_2) \geq 0$ as $Z_1/Z_2 \geq -\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$. Thus for all non-negative values (which are the only economically meaningful values), $Z_1/Z_2$ converges to $\sqrt{\frac{\alpha_2 \theta_2}{\alpha_1 \theta_1}}$.

5 Open Economy Under Complete Specialization

5.1 Basic Setup

The final good production function in the home country is

$$Y_H = X_{H1}X_{H2}^{1-\epsilon}$$

and the production function for processed good 1 is

$$X_{H1} = \int_0^{N_{H1}} (G_{H1j} - G_{H1j}^E)^\lambda \left(Z_{H1}^{(\delta+\gamma)}(Z_{H2}^{(\delta+\gamma)})^{1-(\delta+\gamma)}l_{H1H}\right)^{1-\lambda} dj$$

$$+ \int_0^{N_{F1}} (G_{F1j})^\lambda \left(Z_{F1}^{(\delta+\gamma)}(Z_{H2}^{(\delta+\gamma)})^{1-(\delta+\gamma)}l_{H1F}\right)^{1-\lambda} dj$$

where $G_{H1}^E$ is the amount of $G_H$ produced in the home country and exported, $G_{F1}^I$ is the amount of good $G_F$ produced abroad and imported, $l_{H1H}$ and $l_{H1F}$ are the amounts of labor in the home country making $X_{H1}$ using $G_H$ and $G_F$, respectively, and $\hat{Z}_{H2}$ is the quality spillover whose value depends on the goods being used:

$$\hat{Z}_{H2} = \begin{cases} Z_{H2} & \text{if industry 2 chooses domestic good} \\ Z_{H2}^\eta Z_{F2}^{1-n} & \text{if industry 2 chooses both goods} \\ Z_{F2} & \text{if industry 2 chooses foreign good} \end{cases}$$

By following similar steps to those in Section 1 to maximize the profit of processed-good producer $X_{H1}$ given the prices of intermediate goods and wage, we get the demand function for the domestic good, $(G_{H1j} - G_{H1j}^E)$, and the foreign good, $G_{F1j}^I$. Set the final good of home country as numeraire:

$$P_{Y_H} = 1$$

Denote the price of the domestically produced intermediate good as $P_{G_{H1}}$ and price of the foreign good as $P_{G_{F1}}$. Maximizing profit of the processed good firm, given the prices of intermediate goods and wages, we get

$$P_{X_{H1}}[(\lambda(G_{H1j} - G_{H1j}^E))^\lambda \left(Z_{H1}^{(\delta+\gamma)}(Z_{H2}^{(\delta+\gamma)})^{1-(\delta+\gamma)}l_{H1H}\right)^{1-\lambda} = P_{G_{H1}}$$

so the demand function for domestic good is

$$G_{H1j} - G_{H1j}^E = \left[\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}}ight]^{\frac{1}{\lambda}} Z_{H1}^{(\delta+\gamma)}(Z_{H2}^{(\delta+\gamma)})^{1-(\delta+\gamma)}l_{H1H}$$

(77)
Similarly, we get the demand for foreign good:

\[ G_{F1j} = \left( \frac{\lambda P_{XH1}}{P_{G1F}} \right)^{\frac{1}{\gamma + \delta}} Z_{F1}^{\delta + \gamma} (Z_{H2})^{1 - (\delta + \gamma)} l_{H1F} \]  \hspace{1cm} (78)

The demands for workers are,

\[ l_{H1H} = (P_{XH1} - \frac{1 - \lambda}{w}) \frac{i}{Z} (G_{H1j} - G_{E1j}) (Z_{H1j} Z_{H1} (Z_{H2})^{1 - (\delta + \gamma)})^{\frac{1 - \lambda}{\lambda}} \]  \hspace{1cm} (79)

\[ l_{H1F} = (P_{XH1} - \frac{1 - \lambda}{w})^\frac{i}{Z} G_{F1j} (Z_{F1j} Z_{F1} (Z_{H2})^{1 - (\delta + \gamma)})^{\frac{1 - \lambda}{\lambda}} \]  \hspace{1cm} (80)

Substitute the demand for labor 80 into 78 and note that the \( G_i \)'s on both sides cancel, giving

\[ G_{F1j} = \left( \frac{\lambda P_{XH1}}{P_{G1F}} \right)^{\frac{1}{\gamma + \delta}} Z_{F1}^{\delta + \gamma} (Z_{H2})^{1 - (\delta + \gamma)} l_{H1H} \]  \hspace{1cm} (78)

\[ = \left( \frac{\lambda P_{XH1}}{P_{G1F}} \right)^{\frac{1}{\gamma + \delta}} Z_{F1}^{\delta + \gamma} (Z_{H2})^{1 - (\delta + \gamma)} (P_{XH1} - \frac{1 - \lambda}{w})^\frac{i}{Z} G_{F1j} (Z_{F1j} Z_{F1} (Z_{H2})^{1 - (\delta + \gamma)})^{\frac{1 - \lambda}{\lambda}} \]  \hspace{1cm} (79)

\[ \Rightarrow w = P_{XH1} [\lambda P_{XH1}]^{\frac{1}{\gamma + \delta}} (\frac{1}{P_{G1H}})^{\frac{1}{\gamma + \delta}} (1 - \lambda) Z_{F1}^{\delta + \gamma} (Z_{H2})^{1 - (\delta + \gamma)} \]  \hspace{1cm} (81)

where \( w \) is the wage if the processed good firms use \( G_{F1j} \). Similarly, substituting 79 into 77 gives

\[ w = P_{XH1} [\lambda P_{XH1}]^{\frac{1}{\gamma + \delta}} (\frac{1}{P_{G1H}})^{\frac{1}{\gamma + \delta}} (1 - \lambda) Z_{F1}^{\delta + \gamma} (Z_{H2})^{1 - (\delta + \gamma)} \]  \hspace{1cm} (82)

Rearranging 81 and 82, we get

\[ w_{H1H} = \Upsilon (Z_{H1}^{\frac{1 - \lambda}{P_{G1H}}} P_{G1H})^{\frac{1}{\gamma + \delta}} \]  \hspace{1cm} (83)

\[ w_{H1F} = \Upsilon (Z_{F1}^{\frac{1 - \lambda}{P_{G1F}}} P_{G1F})^{\frac{1}{\gamma + \delta}} \]

where \( \Upsilon = P_{XH1} (1 - \lambda) [\lambda P_{XH1}]^{\frac{1}{\gamma + \delta}} (Z_{H2}^{1 - (\delta + \gamma)}) \).

From 83, given the spillover level \( Z_{H2} \), the processed good producers in industry 1 choose the intermediate good 1 that gives the worker a higher marginal product. Processed-good producers \( X_1 \) do the same: given the spillover from industry 1, which is \( Z_{H1} \), firms choose the intermediate good 2 which give the worker who are using them a higher marginal product. When producers of \( X_2 \) make decisions about \( G_2 \), what they care about is the quality-adjusted price. They ignore the externality they bring to industry 1, which is \( Z_{H2} \). That externality is discussed in more detail in section (5.4).

Note that the intermediate good firms always set the prices \( P_{G1H} \), and \( P_{G1F} \), equal to unit cost multiplied by a markup\(^2\). The prices have nothing to do with the individual firm’s market size because the demand function is constant elasticity. See Microeconomic Analysis (Third Edition) by Hal R. Varian, Section 14.1.

Recall that in the closed economy, we used this way to prove symmetry among firms on the same industry. The wage is \( W_i = (1 - \lambda) (\lambda/P_i)^{\lambda/(1 - \lambda)} Z_i^{1 - \delta} \). Every firm in the industry sets the price at \( A_i/\lambda \), so the only possible difference in the wage is the quality level, \( Z_i \). If a firm has a higher quality, \( Z_i > Z_j \), then \( W_i > W_j \) and that firm will take over the whole market, become a monopoly, and earn a positive profit. Because entry cost is zero, new firms enter instantaneously with the average quality of the incumbent (by assumption), so the equilibrium is that all firms in the same industry have the same quality level.

\(^2\) \( P_{G2} \) needs to be rescaled by the final good price of foreign country.
5.2 Define the Trade Pattern

International trade takes place if each country has a lower (or equal) quality-adjusted price for one intermediate good. For concreteness and with no loss of generality, we assume throughout that the home country has a lower quality-adjusted price for $G_1$ goods and the foreign country has a lower quality-adjusted price for $G_2$ goods:

$$\frac{P_{G_{H1}}}{Z_{H1}^{\lambda}} \leq \frac{P_{G_{F1}}}{Z_{F1}^{\lambda}} \quad \text{and} \quad \frac{P_{G_{H2}}}{Z_{H2}^{\lambda}} \geq \frac{P_{G_{F2}}}{Z_{F2}^{\lambda}} \quad (84)$$

The numeraire is the final good in Home, so the price of the final good in Foreign is $P_{Y_F}$. Thus above condition becomes

$$\frac{A_{H1}/\lambda}{Z_{H1}^{\lambda}} \leq \frac{P_{Y_F} A_{F1}/\lambda}{Z_{F1}^{\lambda}} \quad \text{and} \quad \frac{A_{H2}/\lambda}{Z_{H2}^{\lambda}} \geq \frac{P_{Y_F} A_{F2}/\lambda}{Z_{F2}^{\lambda}} \quad (85)$$

When strict inequality holds, each economy is in a state of complete specialization, producing only one class of intermediate good and importing the other from its trading partner. In this section, we discuss complete specialization. We discuss incomplete specialization in section (6).

5.2.1 Home Country: Final Good and Processed Goods

Under complete specialization, Home’s production function for processed good $X_1$ simplifies to

$$X_{H1} = \int_0^{N_{H1}} (G_{H1j} - G_{H1})^\lambda [Z_{H1}^{(\delta + \gamma)} Z_{F2}^{(\delta + \gamma)}]^{-1-\lambda} dj$$

because only Home produces intermediate goods $G_{1j}$. The amount of intermediate good 1 used by Home is $G_{H1j} - G_{H1}$. The amount exported, $G_{F1j}^{E}$, equals the demand for $G_1$-type goods from Foreign. Home does import $G_2$-type goods from Foreign for use in Home’s $X_2$ industry. The spillover in the $X_1$ industry associated with those goods is $Z_{F2}$.

Home’s production function for processed good $X_2$ is

$$X_{H2} = \int_0^{N_{F2}} (G_{H2j}^{E})^\lambda [Z_{F2}^{(\delta + \gamma)}] Z_{H1}^{(1-\delta - \gamma)} [Z_{F2}^{(1-\delta - \gamma)}]^{-1-\lambda} dj$$

Home uses only $G_2$-type goods imported from Foreign. The number of varieties of such goods is $N_{F2}$.

We already have proven symmetry for firms inside a given industry, and that proof carries over here. Consequently, we drop the subscript $j$. Following exactly the same steps as in section (1), we get the demand functions

$$G_{H1} - G_{H1}^E = \left[\frac{\lambda (X_{H1})^{\epsilon - 1}}{P_{G_{H1}}}\right] \frac{1}{\epsilon} Z_{H1j}^\delta Z_{H1}^{1-\gamma} Z_{H1}^{1-\delta - \gamma} l_{H1} \quad (86)$$

$$G_{H2}^E = \left[\frac{\lambda (1 - \epsilon)(X_{H2})^{\epsilon}}{P_{G_{H2}}}\right] \frac{1}{\epsilon} Z_{F2j}^\delta Z_{F2}^{1-\gamma} Z_{H1}^{1-\delta - \gamma} l_{H2} \quad (87)$$

The trade balance condition is

$$N_{H1} P_{G_{H1}} G_{H1}^E = N_{F2} P_{G_{F2}} G_{H2}^E$$
5.2.2 Foreign Country: Final Good and Processed Goods

The final good production function in Foreign is

\[ Y_F = X_{F1}^{1-\epsilon} X_{F2}^{1-\epsilon} \]  (88)

and the processed goods production functions are

\[ X_{F1j} = \int_0^{N_H} (G_{F1j})^{\lambda} |Z_{H1j}^{\gamma} Z_{F2j}^{\delta} Z_{H1}^{1-(\delta+\gamma)} l_{F1}^{1-\lambda} dj \]  (89)

where the number of varieties of \( G_1 \) available in Foreign is \( N_H \), which is decided by Home, and

\[ X_{F2j} = \int_0^{N_H} (G_{F2j} - F_{F2j})^{\lambda} |Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{F2}^{1-\lambda} dj \]  (90)

Symmetry holds in Foreign by the same arguments as in Home, so again we drop the subscript \( j \).

Final good producers’ profit in Foreign is, in units of the numeraire (Home’s final good \( Y_H \))

\[ \pi_{Y_F} = P_{Y_F} Y_F - P_{X_{F1}} X_{F1} - P_{X_{F2}} X_{F2} \]

The inverse demand functions are

\[ P_{X_{F1}} = P_{Y_F} X_{F1}^{1-\epsilon} \]
\[ P_{X_{F2}} = P_{Y_F} (1-\epsilon) X_{F1}^{1-\epsilon} X_{F2}^{1-\epsilon} \]

Following similar steps as in Home, we get the demand functions of intermediate goods:

\[ G_{F1j}^{I} = \left[ \frac{\lambda \epsilon (X_{F1}^{1-\epsilon})^{\gamma-1} P_{Y_F}}{P_{G_{H1}}} \right] \frac{1}{\epsilon} |Z_{H1j}^{\gamma} Z_{F2j}^{\delta} Z_{H1}^{1-(\delta+\gamma)} l_{F1}^{1-\lambda} \]  (91)

\[ G_{F2j}^{E} - G_{F2j}^{E} = \left[ \frac{\lambda (1-\epsilon) (X_{F1}^{1-\epsilon})^{\gamma} P_{Y_F}}{P_{G_{F2}}} \right] \frac{1}{\epsilon} |Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{F2}^{1-\lambda} \]  (92)

The trade balance condition is

\[ N_{H1} P_{G_{H1}} G_{H1}^{E} = N_{F2} P_{G_{F2}} G_{F2}^{I} \]  (93)

We also have the market equilibrium conditions

\[ G_{F1j}^{I} = G_{H1}^{E} \]
\[ G_{F2j}^{E} = G_{H2}^{E} \]

Following exactly the same steps as in section (1) and (2), we also get

\[ \frac{X_{H1}}{X_{F2}} = \frac{X_{F1}}{X_{F2}} = \frac{\epsilon}{1-\epsilon} \left( \frac{P_{G_{H1}}}{P_{G_{F2}}} \right)^{\lambda} \frac{Z_{H1j}^{\gamma} Z_{F2j}^{\delta} Z_{H1}^{1-(\delta+\gamma)}}{N_{H1}} \]  (94)

Adding 86 to 91, we get the total demand for \( G_{H1} \):

\[ G_{H1j} = (G_{H1j} - G_{H1j}^{E}) + G_{F1j}^{I} \]
\[ = (G_{H1j} - G_{H1j}^{E}) + G_{H1j}^{E} \]
\[ = \left[ \frac{\lambda \epsilon (X_{H2}^{1-\epsilon})^{\gamma-1}}{P_{G_{H1}}} \right] \frac{1}{\epsilon} |Z_{H1j}^{\gamma} Z_{F2j}^{\delta} Z_{H1}^{1-(\delta+\gamma)} L_{H1}^{1-\lambda} + \frac{P_{Y_F}}{N_{H1}} L_{F1} \]  (94)

Adding 87 to 92, we get the total demand for \( G_{F2} \):

\[ G_{F2j} = \left[ \frac{\lambda \epsilon (X_{F1}^{1-\epsilon})^{\gamma-1}}{P_{G_{F2}}} \right] \frac{1}{\epsilon} |Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} L_{H2}^{1-\lambda} + \frac{P_{Y_F}}{N_{F2}} L_{F2} \]  (95)
We derive the solution for $P_Y$ in section (5.3) below.

Comparing 94 and 95 with the closed economy demand 11 and 13, we see that each firm now not only faces the domestic market but also an international market. In first-generation trade models, a higher population means a higher market size and hence induces firms to do R&D. Here, in contrast, the individual firm faces not the total population size but rather the individual market size, which is the population size divided by the number of firms. The number of firms changes endogenously, so an increase in effective population due to trade does not cause a higher growth rate. Any effect of trade on growth in this model does not arise through a scale effect.

### 5.2.3 Intermediate Goods Firms

Using the demand functions 94 and 95, along with the R&D production functions $Z_{H1} = \alpha_{H1} R_{H1}$ and $Z_{F2} = \alpha_{F2} R_{F2}$, we follow exactly the same steps as in section (2) to obtain the following.

1. **Home Country:**

   Home intermediate producers set their price as a markup over unit cost:
   
   $$P_{G_{H1}} = A_{H1}/\lambda$$

   The marginal revenue product of their R&D is
   
   $$\frac{\partial F_{H1}}{\partial Z_{H1}} = \delta A_{H1} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 \epsilon}{A_{H1}} \right]^{\frac{1}{1-\epsilon}} \left( \frac{P_{G_{H1}}}{P_{G_{F2}}} \right)^{\frac{\lambda (\epsilon - 1)}{\lambda - \epsilon}} \left( \frac{Z_{H1}}{Z_{F2}} \right)^{1-\gamma} L_{H1} + \frac{P_Y}{\lambda} L_{F1}$$

   The rate of return to Home R&D is
   
   $$r_{H1} = \frac{\partial F_{H1}}{\partial Z_{H1}} \alpha_{H1}$$

   The rate of growth of the Home intermediate good’s quality $Z_{H1}$ is

   $$g_{H1} = \frac{\dot{Z}_{H1}}{Z_{H1}} = \frac{\alpha_{H1} R_{H1}}{Z_{H1}} = \frac{\alpha_{H1} F_{H1}}{Z_{H1}}$$

   $$= \alpha_{H1} \left\{ A_{H1} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 \epsilon}{A_{H1}} \right]^{\frac{1}{1-\epsilon}} \left( \frac{P_{G_{H1}}}{P_{G_{F2}}} \right)^{\frac{\lambda (\epsilon - 1)}{\lambda - \epsilon}} \left( \frac{Z_{H1}}{Z_{F2}} \right)^{1-\gamma} L_{H1} + \frac{P_Y}{\lambda} L_{F1} + \frac{P_{G_{H1}}}{P_{G_{F2}}} \left( \frac{Z_{H1}}{Z_{F2}} \right)^{1-\gamma} (l_{H2} P_{Y_f} + l_{F2}) \right\}$$

2. **Foreign Country**

   Foreign intermediate producers also set their price as a markup over unit cost:

   $$P_{G_{F2}} = P_{Y_f} A_{F2}/\lambda$$

   The marginal revenue product of Foreign R&D is

   $$\frac{\partial F_{F2}}{\partial Z_{F2}} = P_{F2} \delta A_{F2} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_{F2}} \right]^{\frac{1}{1-\epsilon}} \frac{P_{G_{H1}}}{P_{G_{F2}}} \left( \frac{Z_{H1}}{Z_{F2}} \right)^{1-\gamma} L_{H1} + \frac{P_Y}{\lambda} L_{F1}$$

   and the rate of return to R&D is

   $$r_{F2} = \frac{\partial F_{F2}}{\partial Z_{F2}} / q_{F2}; \quad q_{F2} = \frac{P_{Y_f}}{\alpha_{F2}}$$

\[\text{for} \quad l_{H2} P_{Y_f} + l_{F2} \equiv P_{Y_f} \frac{l_{H1}}{N_{F2}} + \frac{L_{F2}}{N_{F2}}\]
The rate of growth of Foreign quality $Z_{F2}$ is

$$g_{F2} = \frac{\dot{Z}_{F2}}{Z_{F2}} = \frac{\alpha_{F2} R_{F2}}{Z_{F2}} = \frac{\alpha_{F2} F_{F2}/P_{Y_F}}{Z_{F2}}$$

$$= \alpha_{F2} \{ A_{F2} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_{F2}} \right] \frac{\varphi(x)}{1 - \epsilon} \left[ \frac{P_{GH1}}{P_{GF2}} \right] \frac{Z_{F2}}{Z_{H1}} \} \frac{\varphi(x)}{(l_{H2} P_{Y_F}^{\frac{1}{\gamma}} + l_{F2})(\frac{Z_{F2}}{Z_{H1}})^\gamma - \theta_{F2} \frac{Z_{F2}}{Z_{H1}}}$$

Note that in $r_{F2}$, the market size by each firm we express as $l_{H2} P_{Y_F}^{\frac{1}{\gamma}} + l_{F2}$, but in 95, it was $l_{H2} + l_{F2} P_{Y_F}^{\frac{1}{\gamma}}$. The difference arises from using $P_{Y_F}$ to convert to units of the Home final good. The details of the calculations follow the same steps as in section (2.2). The current value Hamiltonian

$$CVH_{F2} = G_{F2}(P_{GF2} - P_{Y_F} A_{F2}) - P_{Y_F} \theta_{F2} \frac{Z_{F2}^2}{Z_{H1}} - P_{Y_F} R_{F2} + q_{2F}(\alpha_{F2} R_{F2})$$

is maximized s.t. the demand function 95 and the R&D production function $\dot{Z}_{F2} = \alpha_{F2} R_{F2}$:

$$\frac{\partial CVH_{F2}}{\partial P_{GF2}} = 0 \Rightarrow P_{GF2} = P_{Y_F} A_{F2}/\lambda$$

$$\frac{\partial CVH_{F2}}{\partial R_{F2}} = -P_{Y_F} + q_{2F} \alpha_{F2} \Rightarrow \text{Interior solution: } q_{2F} = \frac{P_{Y_F}}{\alpha_{F2}}$$

$$F_{F2} = G_{F2}(P_{GF2} - P_{Y_F} A_{F2}) - \theta_{F2} \frac{Z_{H1} + Z_{F2}}{2}$$

Substituting 95 into the last equation gives

$$F_{F2j} = P_{Y_F} A_{F2} \left( \frac{1}{\lambda} - 1 \right) \left\{ \frac{\lambda(1 - \epsilon)}{\lambda} \right\} \frac{\varphi(x)}{1 - \epsilon} \left[ \frac{P_{GH1}}{P_{GF2}} \right] \frac{Z_{H1}}{Z_{F2}}$$

$$\bullet [Z_{F2j}^\frac{\delta}{\gamma} Z_{F2j}^\frac{\gamma}{\gamma} (l_{H2} + P_{Y_F}^{\frac{1}{\gamma}} l_{F2})] - P_{Y_F} \theta_{F2} \frac{Z_{F2}^2}{Z_{H1}}$$

$$= P_{Y_F} \{ A_{F2} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_{F2}} \right] \frac{\varphi(x)}{1 - \epsilon} \left[ \frac{P_{GH1}}{P_{GF2}} \right] \frac{Z_{H1}}{Z_{F2j}} \}$$

$$\bullet [Z_{F2j}^\frac{\delta}{\gamma} Z_{F2j}^\frac{\gamma}{\gamma} (l_{H2} + P_{Y_F}^{\frac{1}{\gamma}} l_{F2})] - \theta_{F2} \frac{Z_{F2}^2}{Z_{H1}}$$

$$= P_{Y_F} \{ A_{F2} \frac{1 - \lambda}{\lambda} \left[ \frac{\lambda^2 (1 - \epsilon)}{A_{F2}} \right] \frac{\varphi(x)}{1 - \epsilon} \left[ \frac{P_{GH1}}{P_{GF2}} \right] \frac{Z_{H1}}{Z_{F2j}} \}$$

The total market size is the sum of the domestic market and the foreign market, $l_{H2} + l_{F2} P_{Y_F}^{\frac{1}{\gamma}}$. The price is $P_{Y_F} A_{F2} (\lambda - 1)/\lambda (\epsilon - 1)$. After converting to units of Home final output, these expressions become $l_{H2} P_{Y_F}^{\frac{1}{\gamma}} + l_{F2}$ and $A_{F2}/\lambda$.

$$l_{H2} + l_{F2} P_{Y_F}^{\frac{1}{\gamma}} = \frac{l_{H2} + P_{Y_F}^{\frac{1}{\gamma}} l_{F2}}{N_{F2}}$$
5.3 Trade Balance Condition under Complete Specialization: \( P_{Y_F} = \frac{1 - \gamma L_H}{\epsilon L_F} \)

As we mentioned in section (2.1), the final good producer in Home pays compensation \((1 - \epsilon)\lambda Y_H\) to the intermediate producers of industry 2, which now are Foreign firms. Similarly, the final good producer in Foreign pays compensation \(\epsilon \lambda P_{Y_F} Y_F\), measured with the final good price from Home, to the intermediate producers of industry 1, which are Home firms. Trade balance requires

\[
(1 - \epsilon)\lambda Y_H = \epsilon \lambda P_{Y_F} Y_F
\]  
(98)

The final good production of Home is

\[
Y_H = \kappa_H Z_{H1}^{1 - \gamma} Z_{F2}^{1 - \Gamma} L_H
\]  
(99)

where

\[
\kappa_H = \lambda^{1 - \lambda} (1 - \epsilon)^{\lambda(1 - \gamma)} e^{\lambda(1 - \gamma)} P_{G_{H1}}^{\lambda} P_{G_{H2}}^{\lambda(1 - \gamma)} \epsilon^\gamma (1 - \epsilon)^{1 - \gamma}
\]

Final good production of Foreign is scaled by the relative final good price, \(P_{Y_F}\). Following the same step as in the derivation of 52, we combine 91, 92, 89, and 90 with the production function of foreign final good 88 to get

\[
Y_F = X_{F1} X_{F2}^{1 - \gamma}
\]

\[
= (\kappa_{F1} P_{Y_F}^{\lambda} Z_{H1}^{\lambda(1 - \gamma) + [1 - 2(\lambda + \gamma)] \lambda (1 - \gamma)} Z_{F2}^{\lambda(1 - \gamma) - [1 - 2(\lambda + \gamma)] \lambda (1 - \gamma)} (L_F)^{\lambda(1 - \gamma)} (1 - \epsilon) L_F)^{1 - \gamma}
\]

\[= \kappa_F Z_{H1}^{1 - \gamma} L_{F2}^{1 - \Gamma} (L_F)^{P_{Y_F}^{\lambda}}
\]  
(100)

where

\[
\kappa_F = \lambda^{1 - \lambda} (1 - \epsilon)^{\lambda(1 - \gamma)} e^{\lambda(1 - \gamma)} P_{G_{H1}}^{\lambda} P_{G_{H2}}^{\lambda(1 - \gamma)} \epsilon^\gamma (1 - \epsilon)^{1 - \gamma}
\]

and \(P_{G_{F2}} = P_{Y_F} A_{F2}/\lambda\). Substitute 100 and 99 into 98 to get

\[
(1 - \epsilon) L_H = \epsilon P_{Y_F} L_F P_{Y_F}^{\lambda}
\]

and

\[
P_{Y_F} = \frac{(1 - \epsilon) L_H}{\epsilon L_F} \lambda
\]  
(101)

We can also get 101 by calculating 93, which is much more complicated, and omitted here.

5.4 Level Effect When Trade Opens

5.4.1 Externality

Recall that we have assumed that

\[
\left(\frac{Z_{H1}^{\lambda}}{P_{G_{H1}}}\right)^{\delta(1 - \gamma)} \leq \left(\frac{Z_{F1}^{\lambda}}{P_{G_{F1}}}\right)^{\delta(1 - \gamma)} \leq \left(\frac{Z_{F2}^{\lambda}}{P_{G_{F2}}}\right)^{\delta(1 - \gamma)}
\]

which implies

\[
\frac{P_{G_{H1}}}{Z_{H1}^{\lambda}} \leq \frac{P_{G_{F1}}}{Z_{F1}^{\lambda}} \quad \text{and} \quad \frac{P_{G_{H2}}}{Z_{H2}^{\lambda}} \geq \frac{P_{G_{F2}}}{Z_{F2}^{\lambda}}
\]

Home produces domestic intermediate good 1 and imports intermediate good 2. From section (5.1), if the Home \(X_2\) industry uses the foreign intermediate good 2, \(G_{F2}\), then there is a spillover to the
first industry of $\tilde{Z}_{H2} = Z_{F2}$. If $Z_{F2} > Z_{H2}$, where $Z_{H2}$ is the autarky quality level in intermediate good 2, then importing the foreign intermediate good 2 has a positive externality from industry 2 to industry 1. If instead $Z_{F2} < Z_{H2}$, importing the foreign intermediate good in industry 2 will have a negative externality on industry 1.

Rewrite 52, Home’s final good production function under autarky, as

$$Y_{H}^{Autarky} = \kappa' Z_{H1}^{(\beta+\gamma)e+1-(\beta+\gamma)(1-e)} Z_{H2}^{1-(\beta+\gamma)(1-e)} P_{G1}^{\lambda - \gamma} P_{G2}^{\lambda(1-e)} (\epsilon L_H)^{e+\gamma} (1-e)L_H$$

$$= \kappa' \left[ \frac{Z_{H1}^{\beta+\gamma}}{P_{G1}} \right] Z_{H2}^{1-(\beta+\gamma)e} \epsilon L_H \left[ \frac{Z_{H2}^{\beta+\gamma}}{P_{G2}} \right] Z_{H1}^{1-(\beta+\gamma)(1-e)} (1-e)L_H^{1-e}$$

Where

$$\kappa' = \lambda \frac{\epsilon}{1-\epsilon} \frac{L_H}{L}$$

From the derivation of 52, we see that

$$\left[ \frac{Z_{H1}^{\beta+\gamma}}{P_{G1}} \right] Z_{H2}^{1-(\beta+\gamma)e} \epsilon L_H$$

is the contribution from $X_{H1}$ to $Y_{H}$, in which $Z_{H2}$ is the spillover from $X_{H2}$ and $\epsilon L_H$ is the total labor employed in industry 1. Similarly,

$$\left[ \frac{Z_{H2}^{\beta+\gamma}}{P_{G2}} \right] Z_{H1}^{1-(\beta+\gamma)(1-e)} (1-e)L_H^{1-e}$$

is the contribution from $X_{H2}$ to $Y_{H}$, in which $Z_{H1}$ is the spillover from $X_{H1}$, and $(1-e)L_H$ is the total labor employed in industry 2.

After trade opens, Home continues producing the domestic intermediate good 1 but imports intermediate good 2. Home’s final good production function becomes

$$Y_{H}^{Trade} = \kappa' \left[ \frac{Z_{H1}^{\beta+\gamma}}{P_{G1}} \right] Z_{F2}^{1-(\beta+\gamma)e} \epsilon L_H \left[ \frac{Z_{F2}^{\beta+\gamma}}{P_{G2}} \right] Z_{H1}^{1-(\beta+\gamma)(1-e)} (1-e)L_H^{1-e}$$

(103)

Comparing 103 with 102, we see that imports affect final output through two channels. The first channel is the quality-adjusted price:

$$\frac{P_{G2}}{Z_{F2}^{\beta+\gamma}} = \left[ \frac{P_{G2}}{Z_{H2}^{\beta+\gamma}} \right] \frac{1}{L_H}$$

Imports have a lower quality-adjusted price than the corresponding domestically produced goods, so the quality per price is higher:

$$\frac{Z_{F2}^{\beta+\gamma}}{P_{G2}} > \frac{Z_{H2}^{\beta+\gamma}}{P_{G2}}$$

(104)

This channel has a positive effect on output. The other channel is the spillover (externality) to industry 1, $Z_{F2}^{1-(\beta+\gamma)e}$. If $Z_{F2} > Z_{H2}$, then the externality is positive and increases final output through the first term in 103. If $Z_{F2} < Z_{H2}$, the externality is negative and reduces final output. When the processed-good firms in industry 2 chooses between foreign and domestic intermediate goods, they do not take the externality into account. They only care which goods give them a higher marginal product of the labor that uses that good, but they do not take into account that the lower $Z_{F2}$ enters industry 1 as a negative externality. If the externality is strong enough, total final output can decrease because of this negative externality.
5.4.2 Parameter Conditions

We derive the necessary and sufficient condition for the level of output to fall when trade opens. We then use that result to derive a simple sufficient condition for output to increase when trade opens.

For the level of output to fall when trade opens, we must have

\[ Y_{Trade}^{H} < Y_{Autarky}^{H} \]

We can see from 102 and 103 that

\[ Y_{Trade}^{H} < Y_{Autarky}^{H} \iff \left( \frac{Z_{F2}^{(\delta+\gamma)}}{P_{G,F2}} \right)^{1-\varepsilon} \left( \frac{Z_{H2}^{1-(\delta+\gamma)}}{P_{G,H2}} \right)^{-\varepsilon} < \left( \frac{Z_{H2}^{\delta+\gamma}}{P_{G,H2}} \right) \]

\[ \iff \frac{Z_{H2}}{Z_{F2}} > \left( \frac{P_{G,H2}}{P_{G,F2}} \right)^{(1-\lambda)(\delta+\gamma)-2(\delta+\gamma)} \] (106)

Equation 105 is the necessary and sufficient condition for \( Y \) to fall when trade opens. Equation 106 is an alternative form that is useful below. Recall that we are assuming that Home specializes in type-1 goods and Foreign specializes in type-2 goods, a trade pattern that requires that inequality 84 be satisfied. That inequality implies that the first term on the left side of 105 is greater than 1:

\[ \left( \frac{Z_{F2}^{(\delta+\gamma)}}{P_{G,F2}} \right)^{1-\varepsilon} \left( \frac{Z_{H2}^{1-(\delta+\gamma)}}{P_{G,H2}} \right)^{-\varepsilon} > 1 \]

That result together with 105 then implies that

\[ \frac{Z_{F2}}{Z_{H2}} < 1 \]

\[ \iff Z_{F2} < Z_{H2} \] (107)

Therefore a sufficient condition for \( Y \) to rise when trade opens is

\[ Z_{F2} > Z_{H2} \]

The second inequality in 84 can be written

\[ \left( \frac{P_{G,H2}}{P_{G,F2}} \right)^{(1-\lambda)(\delta+\gamma)-2(\delta+\gamma)} > \left( \frac{Z_{H2}}{Z_{F2}} \right) \]

Combining 108 and 106 gives the interval within which the quality ratio \( Z_{H2}/Z_{F2} \) must fall for trade to have the pattern we have assumed and to reduce current output when trade opens:

\[ \left( \frac{P_{G,H2}}{P_{G,F2}} \right)^{(1-\lambda)(\delta+\gamma)-2(\delta+\gamma)} > \frac{Z_{H2}}{Z_{F2}} \]

\[ \iff \left( \frac{P_{G,H2}}{P_{G,F2}} \right)^{(1-\lambda)(\delta+\gamma)-2(\delta+\gamma)} > \left( \frac{Z_{H2}}{Z_{F2}} \right) \] (109)
5.5 The World Balanced Growth Rate

We have an integrated economy with nine unknowns and nine equations as in section (4.1.1) plus the trade balance condition, which is solved in section (5.3) to get $P_{Y_F}$. The whole dynamic system reduces to 26, 33, 35, and 36, with all $\alpha_1 \theta_1$ replaced by $\alpha_{H1} \theta_{H1}$ and all $\alpha_2 \theta_2$ replaced by $\alpha_{F2} \theta_{F2}$. The price of $G_1$ is the Home price given by 22, $P_{G_{H1}} = A_{H1}/\lambda$. The price of $G_2$ is the Foreign price given by 30 converted to units of the numeraire, $P_{G_{F2}} = P_{Y_F} A_{F2}/\lambda$. As in the closed economy, prices do not affect growth rate, so the changes in parameters in the prices change nothing of substance. The Euler equation for both countries is the same in the closed economy. Following exactly the same steps as Section (4) to get the balanced growth rate:

$$g^*_H = \frac{\delta}{1-\delta} \sqrt{\alpha_{H1} \theta_{H1} \alpha_{F2} \theta_{F2}} - \frac{1}{1-\delta} \rho$$

(110)

where $(Z_{H1}/Z_{F2})^*$ is given by 67 with $\alpha_1 \theta_1 = \alpha_{H1} \theta_{H1}$ and $\alpha_2 \theta_2 = \alpha_{F2} \theta_{F2}$.

5.6 Market Size After Trade

Following the same steps as in section (4.1), we get

$$L_{H1} = \epsilon L_H; \quad L_{H2} = (1-\epsilon) L_H$$
$$L_{F1} = \epsilon L_F; \quad L_{F2} = (1-\epsilon) L_F$$

so

$$L_{H1} + P_{Y_F}^{1-x} L_{F1} = \epsilon L_H + \{[(1-\epsilon)L_H]\}^{1-x} \epsilon L_F$$
$$= \epsilon L_H + (1-\epsilon)L_H$$
$$= L_H$$

which is the total market size that the intermediate good firms in Home’s industry 1 face. Trade increases the total size for Home’s industry 1 from $\epsilon L_H$ to the whole Home population. Similarly,

$$P_{Y_F}^{1-x} L_{H2} + L_{F2} = \{[(1-\epsilon)L_H]\}^{1-x} (1-\epsilon)L_H + (1-\epsilon)L_F$$
$$= \epsilon L_F + (1-\epsilon)L_F$$
$$= L_F$$

which is the total market size that Foreign’s intermediate good firms in industry 2 face., equal to the whole foreign population.

Continuing to follow the steps in section (4.1), we use 110, the Euler equation 54, and 128 to get the market size for each firm in Home:

$$l^*_{H1} = \frac{L_H}{N_{H1}} = \frac{\delta}{1-\delta} \sqrt{\alpha_{H1} \theta_{H1} \alpha_{F2} \theta_{F2}} - \frac{\delta}{1-\delta} \rho$$

(111)

Recall that autarky market size for each firm is 73:

$$l^*_{H1} = \frac{\epsilon L_H}{N_{H1}} = \frac{\delta}{1-\delta} \sqrt{\alpha_{H1} \theta_{H1} \alpha_{H2} \theta_{H2}} - \frac{\delta}{1-\delta} \rho$$

Trade has led to a replacement of the term $\alpha_{H2} \theta_{H2}$ by $\alpha_{F2} \theta_{F2}$. It is straightforward to show that

$$\frac{\partial l^*_{H1}}{\partial (\alpha_2 \theta_2)} > 0$$
but in general we do not know if $\alpha_{H2}\theta_{H2} \geq \alpha_{F2}\theta_{F2}$, and so we do not know if trade raises or lowers firm size. However, we do see from 111 that there is once again no scale effect, exactly as in autarky, as explained in section (4.1).

### 5.7 Transition Dynamics

Similar to the closed economy, we have

$$r_{H1} = f_{H2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_H}{N_{H1}})$$
$$r_{F2} = f_{F2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_F}{N_{F2}})$$
$$g_{H1} = \frac{f_{H2}(\frac{L_H}{Z_{F2}}, \frac{Z_{H1}}{N_{H1}})}{\delta} - \alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}}$$
$$g_{F2} = \frac{f_{F2}(\frac{L_F}{Z_{F2}}, \frac{Z_{H1}}{N_{F2}})}{\delta} - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}$$

The no-arbitrage condition does not hold across countries because there is no international investment. However, from 99 and 100, we have

$$\frac{\dot{Y}_{H}}{Y_{H}} = \frac{\dot{Y}_{F}}{Y_{F}} = \Gamma \frac{\dot{Z}_{H1}}{Z_{H1}} + (1 - \Gamma) \frac{\dot{Z}_{F2}}{Z_{F2}}$$

We assume identical preference functions in the two countries, so by the Euler equation 54,

$$\frac{c_{H}}{c_{H}} = \frac{c_{F}}{c_{F}} = r_{H1} - \rho = r_{F2} - \rho$$
$$\Leftrightarrow r_{H1} = r_{F2}$$

The transitional dynamics are equalized across the two countries, even without international investment. The transitional dynamics are similar to those in the closed economy:

$$\frac{(\dot{Z}_{H1}/Z_{F2})}{(\dot{Z}_{H1}/Z_{F2})} = g_{H1} - g_{F2}$$
$$= \frac{r_{H1}}{\delta} - \alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}} - (\frac{r_{F2}}{\delta} - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}})$$
$$= -\alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}$$

and

$$\frac{(\dot{Z}_{H1}/Z_{F2})}{(\dot{Z}_{H1}/Z_{F2})} = -\alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}$$

As in the closed economy, the positive steady state of the quality ratio $Z_{H1}/Z_{F2}$ is stable.

### 6 Open Economy under Incomplete Specialization

#### 6.1 Define Comparative Advantage

The condition for our maintained trade pattern is the double inequality 85. Complete specialization occurs when both inequalities are strict. In that case, the solution for $P_{Y_F}$ is given by 101, repeated here:

$$P_{Y_F} = \left[\frac{(1-\epsilon)L_H}{\epsilon L_F}\right]^{1-\lambda}$$
The price $P_{Y_F}$ must fall inside the upper and lower bounds of the double inequality 85 because otherwise both countries would try to export the same good and trade balance would be violated. However, there is no reason that the expression on the right side of 101 need be inside the trade pattern interval. When it is outside that interval, then $P_{Y_F}$ cannot be equal to it. Instead, $P_{Y_F}$ will equal whichever of the two trade pattern bounds is closer to $[(1 - \epsilon)L_H/eL_F]^{1 - \lambda}$. One of the two countries will continue to specialize completely but the other will not specialize and instead will continue to produce both types of intermediate goods. For concreteness and with no loss of generality, we assume that $[(1 - \epsilon)L_H/eL_F]^{1 - \lambda}$ is large enough that

$$
[(1 - \epsilon)L_H/eL_F]^{1 - \lambda} \geq \frac{A_{H2}/Z_{H2}^{(\delta + \gamma)(1 - \lambda)}}{A_{F2}/Z_{F2}^{(\delta + \gamma)(1 - \lambda)}} = P_{Y_F} > \frac{A_{H1}/Z_{H1}^{(\delta + \gamma)(1 - \lambda)}}{A_{F1}/Z_{F1}^{(\delta + \gamma)(1 - \lambda)}}
$$

(112)

All the results we derive in this section hold in mirror image if $[(1 - \epsilon)L_H/eL_F]^{1 - \lambda}$ small enough to be below the smaller bound of the trade pattern interval. Under this assumption, the quality-adjusted price of intermediate good 1 is lower for Home’s product. The quality-adjusted price of intermediate good 2 is the same for both Home and Foreign products. Home produces both types of intermediates, and Foreign specializes in producing intermediate good 2. Home exports intermediate good 1 and imports intermediate good 2. Intuitively, Foreign is not “big” enough to fulfill Home’s demand for good 2, so home country produces good 2 as well as good 1.

### 6.2 Model Setup

#### 6.2.1 Home Country: Final Good and Processed Good

Home’s final good and processed good production functions are

$$
Y_H = X_{H1}^{1 - \epsilon}X_{H2}^\epsilon
$$

$$
X_{H1} = \int_0^{N_{H1}} (G_{H1} - G_{H1})^\lambda [Z_{H1}^{(\delta + \gamma)}(Z_{H2})^{1 - (\delta + \gamma)]}^{1 - \lambda} dj
$$

$$
X_{H2} = \int_0^{N_{H2}} G_{H2}^{\lambda}(Z_{H2}^{(\delta + \gamma)}Z_{H1}^{1 - (\delta + \gamma)}l_{H2H})^{1 - \lambda} dj + \int_0^{N_{F2}} (G_{H2}^{\lambda})(Z_{H2}^{(\delta + \gamma)}Z_{H1}^{1 - (\delta + \gamma)}l_{H2F})^{1 - \lambda} dj
$$

Home exports $G_{H1}^{E}$ units of good 1. Home uses both domestic and foreign goods in industry 2. Denote by $L_{H2H}^I = \int_0^{N_{H2}} l_{H2H}^I dj$ and $L_{H2F}^I = \int_0^{N_{F2}} l_{H2F}^I dj$ the amount of labor in Home using domestic and foreign intermediate good 2, respectively. Following similar steps as in section (1), we get the inverse demands for $X_{H1}$ and $X_{H2}$:

$$
P_{X_{H1}} = \epsilon X_{H1}^{1 - \epsilon}X_{H2}^\epsilon
$$

(113)

$$
P_{X_{H2}} = (1 - \epsilon)X_{H1}^{1 - \epsilon}X_{H2}^\epsilon
$$

(114)

and also the demand functions for intermediate goods and labor:

$$
G_{H1j} - G_{H1j}^E = \left(\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}}\right)^\frac{1}{\lambda} Z_{H1}^{(\delta + \gamma)}(Z_{H2})^{1 - (\delta + \gamma)}
$$

(115)

$$
G_{H2j} = \left(\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}}\right)^\frac{1}{\lambda} Z_{H2}^{(\delta + \gamma)}Z_{H1}^{1 - (\delta + \gamma)}l_{H1H}
$$

(116)

$$
G_{H2F} = \left(\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}}\right)^\frac{1}{\lambda} Z_{H2}^{(\delta + \gamma)}Z_{H1}^{1 - (\delta + \gamma)}l_{H1F}
$$

(117)

$$
l_{H1} = (P_{X_{H1}} - 1 - \frac{\lambda}{W_H})^\frac{1}{2}(G_{H1} - G_{H1}^E)Z_{H1}^{(\delta + \gamma)}(Z_{H2})^{1 - (\delta + \gamma)}
$$

(118)

$$
l_{H2H} = (P_{X_{H2}} - 1 - \frac{\lambda}{W_H})^\frac{1}{2}G_{H2}Z_{H2}^{(\delta + \gamma)}Z_{H1}^{1 - (\delta + \gamma)}
$$

(119)
\[ l_{H2F} = (P_{X_{H2}} \frac{1-\lambda}{W_H})^{\frac{1}{2}} G_{H2}^\frac{\lambda}{3} [Z_{H2}^{(\delta+\gamma)} G_{H2}^{(\delta+\gamma)}]^{\frac{1}{2}} \] (120)

The marginal product of Home labor in the \( X_1 \) industry is

\[
ML_{H1} = P_{X_{H1}} \frac{\partial X_{H1}}{\partial H1} = P_{X_{H1}} (1-\lambda)(G_{H1} - G_{H2}^\frac{\lambda}{3}) [Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)}]^{1-(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}
\]

Using the demand function 115, we get

\[
ML_{H1} = P_{X_{H1}} (1-\lambda) (\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}
\]

\[
\bullet Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}
\]

\[
= P_{X_{H1}} (1-\lambda) (\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}
\]

Similarly, we get

\[
ML_{H2F} = P_{X_{H2}} (1-\lambda) (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{1}{1-\lambda}} Z_{H2}^{(\delta+\gamma)} G_{H2}^{(\delta+\gamma)} l_{H2}^{1-(\delta+\gamma)}
\]

and

\[
ML_{H2H} = P_{X_{H2}} (1-\lambda) (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{1}{1-\lambda}} Z_{H2}^{(\delta+\gamma)} G_{H2}^{(\delta+\gamma)} l_{H2}^{1-(\delta+\gamma)}
\]

All marginal products of labor must be equal in Home: \( ML_{H1} = ML_{H2H} = ML_{H2F} \). The latter equality, \( ML_{H2H} = ML_{H2F} \), is equivalent to our trade pattern condition:

\[
\frac{P_{H2}}{Z_{H2}^{\frac{1}{1-\lambda}}} = \frac{P_{F2}}{Z_{F2}^{\frac{1}{1-\lambda}}}
\]

\[
\iff \frac{A_{H2}}{Z_{H2}^{\frac{1}{1-\lambda}}} = \frac{A_{F2}}{Z_{F2}^{\frac{1}{1-\lambda}}} = P_{F2}
\]

where \( P_{H2} = A_{H2}/\lambda \) and \( P_{F2} = Y_{F} A_{F2}/\lambda \). Domestic and foreign goods yield the same marginal product for labor, so Home chooses both goods. Equality of the three marginal products also implies

\[
\frac{P_{X_{H1}}}{P_{X_{H2}}} = \frac{P_{G_{H1}}}{P_{G_{H2}}} = \frac{Z_{H2}^{(\delta+\gamma)} G_{H2}^{(\delta+\gamma)} l_{H2}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}}
\]

(121)

Now we use \( G_{ij} \) and \( Z_{ij} \) to express \( P_{X_{H1}} \) and \( P_{X_{H2}} \). Divide 113 by 114 and rearrange terms to get \( X_{H1}/X_{H2} \) as a function of \( P_{X_{H1}}/P_{X_{H2}} \):

\[
\frac{P_{X_{H1}}}{P_{X_{H2}}} = \frac{\epsilon X_{H1}}{1-\epsilon X_{H2}}
\]

\[
\iff \frac{X_{H1}}{X_{H2}} = \frac{\epsilon P_{X_{H1}}}{1-\epsilon P_{X_{H2}}}
\]

Next substitute 121 into the right side of this last equation:

\[
\frac{X_{H1}}{X_{H2}} = \frac{\epsilon}{1-\epsilon} \left( \frac{P_{X_{H1}}}{P_{X_{H2}}} \right)^{-1}
\]

\[
= \frac{\epsilon}{1-\epsilon} \left[ \left( \frac{P_{G_{H1}}}{P_{G_{H2}}} \right)^{\frac{1}{1-\lambda}} \frac{Z_{H2}^{(\delta+\gamma)} G_{H2}^{(\delta+\gamma)} l_{H2}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} G_{H1}^{(\delta+\gamma)} l_{H1}^{1-(\delta+\gamma)}} \right]^{-1-\lambda}
\]

30
Substitute this expression back into the inverse demand functions 113 and 114 to get $P_{X_{H1}}$ and $P_{X_{H2}}$:

\[
P_{X_{H1}} = \epsilon \left\{ \frac{\epsilon}{1 - \epsilon} \left[ \left( \frac{P_{G_{H1}}}{P_{G_{H2}}} \right)^{\alpha} \frac{Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)}Z_{H2}^{1-(\delta+\gamma)}} \right]^{-1} \right\}^{\epsilon-1} \tag{122}\]

By the trade balance condition

\[
P_{H2}/Z_{H1}^{(\delta+\gamma)(1-\lambda)} = P_{F2}/Z_{F2}^{(\delta+\gamma)(1-\lambda)}
\]

we can write $P_{X_{H1}}$ as

\[
P_{X_{H1}} = \epsilon \left\{ \frac{\epsilon}{1 - \epsilon} \left[ \left( \frac{P_{G_{H1}}}{P_{G_{H2}}} \right)^{\alpha} \frac{Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)}Z_{H2}^{1-(\delta+\gamma)}} \right]^{-1} \right\}^{\epsilon} \tag{123}\]

Similarly,

\[
P_{X_{H2}} = (1-\epsilon) \left\{ \frac{\epsilon}{1 - \epsilon} \left[ \left( \frac{P_{G_{H1}}}{P_{G_{H2}}} \right)^{\alpha} \frac{Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)}Z_{H2}^{1-(\delta+\gamma)}} \right]^{-1} \right\}^{\epsilon}
\]

Home producers in processed good industry 2 use both domestic and foreign intermediate goods

\[
X_{H2} = \int_{0}^{N_{H2}} G_{H2}^{\lambda}Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2H}^{1-\lambda}dj + \int_{0}^{N_{F2}} (G_{F2}^{\lambda})^{(\delta+\gamma)}Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2F}^{1-\lambda}dj
\]

Denote

\[
X_{H2H} = \int_{0}^{N_{H2}} G_{H2}^{\lambda}Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2H}^{1-\lambda}dj
\]

\[
X_{H2F} = \int_{0}^{N_{F2}} (G_{F2}^{\lambda})^{(\delta+\gamma)}Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2F}^{1-\lambda}dj
\]

Plug the demand functions 116 and 117 inside these expressions and take note of symmetry to get

\[
X_{H2H} = N_{H2}\left( \frac{\lambda P_{X_{H2}}}{P_{G_{H2}}} \right)^{\alpha} (Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2H}^{1-\lambda}Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}l_{H2H}^{1-\lambda}) = \left( \frac{\lambda P_{X_{H2}}}{P_{G_{H2}}} \right)^{\alpha} Z_{H2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}L_{H2H}
\]

where $L_{H2H} = N_{H2}l_{H2H}$ and

\[
X_{H2F} = \left( \frac{\lambda P_{X_{H2}}}{P_{G_{H2}}} \right)^{\alpha} Z_{F2}^{(\delta+\gamma)}Z_{H1}^{1-(\delta+\gamma)}L_{H2F}
\]

where $L_{H2F} = N_{F2}l_{H2F}$. Thus, $X_{H2H}/X_{H2F} = L_{H2H}/L_{H2F}$, given the trade pattern condition $Z_{H2}^{(\delta+\gamma)}/P_{H2}^{1-\lambda} = Z_{F2}^{(\delta+\gamma)}/P_{F2}^{1-\lambda}$. So $X_{H2H} = (L_{H2H}/L_{H2})X_{H2}$ and $X_{H2F} = (L_{H2F}/L_{H2})X_{H2}$, where $L_{H2H} + L_{H2F} = L_{H2}$. Producers using $X_{H2H}$ and $X_{H2F}$ both devote the proportion $\lambda$ of their product to compensation for labor, $L_{H2H}$ and $L_{H2F}$, so the entire $X_{H2}$ industry devotes the proportion $\lambda$ of output to compensate the labor total $L_{H2} = L_{H2H} + L_{H2F}$.

By resource allocation as in section (1), we get

\[
\frac{L_{H1}}{L_{H2H} + L_{H2F}} = \frac{\epsilon}{1 - \epsilon} \tag{124}
\]
We can also write

\[ X_{H2} = \left( \frac{\lambda P_{X_{H2}}}{P_{GH2}} \right)^{\frac{1}{\lambda}} Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} (L_{H2H} + L_{H2F}) \]

\[ = \left( \frac{\lambda P_{X_{H2}}}{P_{GH2}} \right)^{\frac{1}{\lambda}} Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} (L_{H2H} + L_{H2F}) \]

### 6.2.2 Foreign Country: Final Good and Processed Good

Foreign’s production functions are

\[ Y_F = X_{F1} X_{F2}^{1-\epsilon} \]

\[ X_{F1} = \int_0^{N_{H1}} (G_{F1}^I)^{\lambda} [Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}]^{1-\lambda} dj \]

\[ X_{F2} = \int_0^{N_{F2}} (G_{F2} - G_{F2}^E)^{\lambda} [Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}]^{1-\lambda} dj \]

The price of \( Y_F \) is \( P_{Y_F} \). Foreign imports \( G_{F1}^E = G_{F1}^E \) from Home, and exports \( G_{F2}^E = G_{F2}^E \) to Home.

The inverse demand functions for \( X_{F1} \) and \( X_{F2} \) are

\[ P_{X_{F1}} = P_{Y_F} \epsilon X_{F1}^{1-\epsilon} X_{F2} \]

\[ P_{X_{F2}} = P_{Y_F} (1 - \epsilon) X_{F1}^{1-\epsilon} X_{F2} \]

The demands for intermediate goods are

\[ G_{H1}^E = G_{F1}^I = \left( \frac{\lambda P_{X_{F1}}}{P_{GH1}} \right)^{\frac{1}{\lambda}} Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} \]

\[ G_{F2} - G_{F2}^E = \left( \frac{\lambda P_{X_{F2}}}{P_{GF2}} \right)^{\frac{1}{\lambda}} Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} \]

From \( MP_{l_{F1}} = MP_{l_{F2}} \), we get

\[ \left( \frac{P_{X_{F1}}}{P_{X_{F2}}} \right)^{\frac{1}{\lambda}} = \left( \frac{P_{GH1}}{P_{GF2}} \right)^{\frac{1}{\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}} \]

Following the same steps as in the derivation of 122 and 123, we get

\[ P_{X_{F1}} = \epsilon \left\{ \frac{\epsilon}{1 - \epsilon} \left[ \left( \frac{P_{GH1}}{P_{GF2}} \right)^{\frac{1}{\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}} \right]^{1-(1-\lambda)} \right\}^{\epsilon-1} \]  \hspace{1cm} (125)

\[ P_{X_{F2}} = (1 - \epsilon) \left\{ \frac{\epsilon}{1 - \epsilon} \left[ \left( \frac{P_{GH1}}{P_{GF2}} \right)^{\frac{1}{\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}} \right]^{1-(1-\lambda)} \right\}^{\epsilon} \]  \hspace{1cm} (126)

As before, resource allocation implies

\[ \frac{N_{H1} l_{F1}}{N_{F2} l_{F2}} = \frac{L_{F1}}{L_{F2}} = \frac{\epsilon}{1 - \epsilon} \]
6.2.3 Home Country: Intermediate Good 1

The total demand facing Home’s intermediate good firms in industry 1 is

\[ G_{H1} = (G_{H1} - G_{H1}^E) + G_{F1} \]

\[ = \left( \frac{\lambda P_{X_{H1}}}{P_{G_{H1}}} \right) Z_{H1}^{(\delta+\gamma)} (Z_{H2})^{1-(\delta+\gamma)} l_{H1} + \left( \frac{\lambda P_{X_{F1}}}{P_{G_{H1}}} \right) Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1} \]

\[ = P_{G_{H1}}^{-1} (\lambda P_{X_{H1}}) Z_{H1}^{(\delta+\gamma)} (Z_{H2})^{1-(\delta+\gamma)} l_{H1} + (\lambda P_{X_{F1}}) Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1} \]

The cash flow to those firms is

\[ F_{H1} = G_{H1} (P_{G_{H1}} - A_{H1}) - \theta_{H1} \frac{Z_{H1}^2}{Z_{H2}} \]

The firm maximizes its current value Hamiltonian

\[ CV H_{H1} = G_{H1} (P_{G_{H1}} - A_{H1}) - \theta_{H1} \frac{Z_{H1}^2}{Z_{H2}} - R_{H1} + q_{H1} (\alpha_{H1} R_{H1}) \]

s.t. the demand function for \( G_{H1} \), which is 127, and the quality-improvement production function \( Z_{H1} = \alpha_{H1} R_{H1} \). Following similar steps as in section (2), we get the necessary conditions

\[ Z_{H1} = \alpha_{H1} R_{H1} \]

\[ \frac{\partial CV H_{H1}}{\partial P_{G_{H1}}} = 0 \Rightarrow P_{G_{H1}} = \frac{A_{H1}}{\lambda} \]

\[ \frac{\partial CV H_{H1}}{\partial R_{H1}} = -1 + q_{H1} \alpha_{H1} \Rightarrow \begin{cases} q_{H1} < \frac{1}{\alpha_{H1}}, & R_{H1} = 0 \\ q_{H1} = \frac{1}{\alpha_{H1}}, & R_{H1} > 0 \text{ (we focus on this case)} \\ q_{H1} > \frac{1}{\alpha_{H1}}, & R_{H1} = \infty \end{cases} \]

\[ Z_{H1, t=0} \text{ given} \]

\[ \lim_{t \to -\infty} e^{-\int_0^t \text{d}s} q_{H1} (t) Z_{H1} (t) = 0 \]

\[ q_{ij}^* = -r_{ij} q_{ij} + \frac{\partial CV H_{ij}}{\partial Z_{ij}} \]

\[ \Rightarrow r_{H1} = \frac{\partial F_{H1}}{\partial Z_{H1}} / q_{H1} + \frac{q_{H1}}{q_{H1}} \]

It is convenient to write \( r_{H1} \) as

\[ r_{H1} = \delta f_{H1} (\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, N_{H1}) \]

where

\[ f_{H1} = \alpha_{H1} \frac{\lambda P_{X_{H1}}}{A_{H1}} Z_{H1}^{(\delta+\gamma)-1} \left( (\frac{\lambda P_{X_{H1}}}{A_{H1}}) Z_{H2} Z_{H1}^{1-(\delta+\gamma)} l_{H1} + (\frac{\lambda P_{X_{F1}}}{A_{H1}}) Z_{F2} Z_{H1}^{1-(\delta+\gamma)} l_{F1} \right) \]

\[ l_{H1} = \frac{\alpha_{H1}}{A_{H1}} \]

\[ l_{F1} = \frac{\alpha_{H1}}{A_{H1}} \]

\( P_{X_{H1}} \) is a function of unit costs, \( \frac{Z_{H1}}{Z_{H2}} \), and \( \frac{Z_{H1}}{Z_{F2}} \), and is given by 122

\( P_{X_{F1}} \) is a function of unit costs and \( \frac{Z_{H1}}{Z_{F2}} \), and is given by 125

Zero entry cost ensures that the zero profit condition holds, so

\[ g_{H1} = \delta f_{H1} \]

\[ g_{H1} = \alpha_{H1} F_{H1} = \delta f_{H1} (\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, N_{H1}) - \alpha_{H1} \theta_{H1} \frac{Z_{H1}}{Z_{H2}} \]
6.2.4 Home Country: Intermediate Good 2

Firms in Home’s industry 2 produce goods that are used only by Home country. Firms’ cash flow is

\[ F_{H2} = G_{H2}(P_{G_{H2}} - A_{H2}) - \theta_{H2} \frac{Z_{H2}^2}{Z_{H1}} \]

The firm’s problem is to maximize

\[ CV_{H2} = G_{H2}(P_{G_{H2}} - A_{H2}) - \theta_{H2} \frac{Z_{H2}^2}{Z_{H1}} - R_{H2} + q_{H2}(\alpha_{H2}R_{H2}) \]

s.t. the demand function for \( G_{H2} \), which is 116, and the quality-improvement production function \( Z'_{H2} = \alpha_{H2}R_{H2} \). Once again following similar steps as in section (2), we get the necessary conditions:

\[ Z'_{H2} = \alpha_{H2}R_{H2} \]

\[ \frac{\partial CV_{H2}}{\partial P_{G_{H2}}} = 0 \Rightarrow P_{G_{H2}} = \frac{A_{H2}}{\lambda} \]

\[ \frac{\partial CV_{H2}}{\partial R_{H2}} = -1 + q_{H2}\alpha_{H2} \Rightarrow \begin{cases} q_{H2} < \frac{1}{\alpha_{H2}}, & R_{H2} = 0 \\ q_{H2} = \frac{1}{\alpha_{H2}}, & R_{H2} > 0 \ (\text{we focus on this case}) \\ q_{H2} > \frac{1}{\alpha_{H2}}, & R_{H2} = \infty \end{cases} \]

\[ Z_{H2,t=0} \text{ given} \]

\[ \lim_{t \to \infty} e^{-\int_{t}^{\infty} r(s)ds} q_{H2}(t) Z_{H2}(t) = 0 \]

\[ q_{H2} = -r_{H2}q_{H2} + \frac{\partial CV_{H2}}{\partial Z_{H2}} \Rightarrow r_{H2} = \frac{\partial F_{H2}}{\partial Z_{H2}} q_{H2} + \frac{\partial CV_{H2}}{\partial Z_{H2}} \]

As with intermediate good 1, we write \( r_{H2} \) in the simplified form

\[ r_{H2} = \delta f_{H2}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{F2}}{Z_{H2}}, N_{H2}) \quad (130) \]

where

\[ f_{H2} = \alpha_{H2} \frac{\lambda}{\lambda - \lambda_{F2}} \frac{\lambda_{F2}}{Z_{H2}}^{\frac{\lambda_{F2}}{\lambda}} \left( \frac{Z_{H2}}{Z_{H1}} \right)^{(\delta_{H} - \gamma) - 1} l_{H2} \]

\[ l_{H2} = \frac{L_{H2}}{N_{H2}} \]

\[ L_{H2} = (1 - \epsilon) L_{H} - L_{F,H} \]

\( L_{F,H} \) is a function of \( \frac{Z_{H2}}{Z_{F1}} \) by trade balance condition, 138

\( P_{X_{H2}} \) is a function of \( \frac{Z_{H1}}{Z_{F1}} \) and \( \frac{Z_{H2}}{Z_{F2}} \) and is given by 123

Zero entry cost implies that the zero profit condition always is satisfied, so

\[ \frac{\alpha_{H2}R_{H2}}{Z_{H2}} = \frac{\alpha_{H2}F_{H2}}{Z_{H2}} \]

\[ = f_{H2}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{F2}}{Z_{H2}}, N_{H2}) - \alpha_{H2} \theta_{H2} \frac{Z_{H2}}{Z_{H1}} \]

\[ = q_{H2} \]

34
6.2.5 Foreign Country: Intermediate Good 2

Firms in Foreign industry 2 produce goods used by both Home and Foreign. The demand function is

\[ G_{F2} = \left[ \frac{\lambda}{P_{G_{F2}}} \right]^{1-\delta} Z_{F2}^{\frac{1}{\delta}} (Z_{H1})^{1-\delta} (P_{X_{H2}}^{1-\gamma} l_{H2F} + P_{X_{H2}}^{1-\gamma} l_{F2}) \]  

(132)

The cash flow is

\[ F_{F2} = G_{F2}(P_{G_{F2}} - P_{l_{F2}} A_{F2}) - P_{l_{F2}} \theta_{F2} Z_{F2}^2 Z_{H1} \]

The firm maximizes

\[ CVH_{F2} = G_{F2}(P_{G_{F2}} - P_{l_{F2}} A_{F2}) - P_{l_{F2}} \theta_{F2} Z_{F2}^2 Z_{H1} - P_{l_{F2}} R_{F2} + q_{F2}(\alpha_{F2} R_{F2}) \]

s.t. the demand function 132; and the quality-improvement production function \( \dot{Z}_{F2} = \alpha_{F2} R_{F2} \). The necessary conditions are

\[ \frac{\partial CVH_{F2}}{\partial P_{G_{F2}}} = 0 \Rightarrow P_{G_{F2}} = P_{Y_{F2}} \frac{A_{F2}}{X} \]

\[ \frac{\partial CVH_{F2}}{\partial R_{F2}} = -P_{l_{F2}} + q_{F2} \alpha_{F2} \Rightarrow \text{Interior solution:} \ q_{2F} = \frac{P_{Y_{F2}}}{\alpha_{F2}} \]

We write \( r_{F2} \) in the simplified form

\[ r_{F2} = \delta f_{F2} \left( \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{H2}}{Z_{F2}}, \frac{Z_{H1}}{Z_{F1}}, N_{F2} \right) \]

(133)

where

\[ f_{F2} = \alpha_{F2} A_{F2} \frac{1-\delta}{X} \left( \frac{\lambda^2}{P_{l_{F2}} A_{F2}} \right)^{1-\gamma} \left( \frac{Z_{F2}}{Z_{H1}} \right)^{1-\delta} (P_{X_{F2}}^{1-\gamma} l_{F2} + P_{X_{H2}}^{1-\gamma} l_{H2F}) \]

\[ l_{F2} = \frac{(1-\gamma)L_{EF}}{P_{X_{F2}}} \text{ where } L_{EF} \text{ is a function of } \frac{Z_{H1}}{Z_{F2}} \text{ from 138} \]

\[ P_{X_{F2}} \text{ is a function of } \frac{Z_{H1}}{Z_{F2}} \text{ in 126} \]

\[ P_{X_{H2}} \text{ is a function of } \frac{Z_{H1}}{Z_{F2}} \text{ and } \frac{Z_{H2}}{Z_{F2}} \text{ in 123} \]

\[ P_{Y_{F2}} \text{ is a function of } \frac{Z_{H1}}{Z_{F2}} \text{ by the trade pattern condition 85} \]

Free entry implies that the zero profit condition always holds, so

\[ \frac{\alpha_{F2} R_{F2}}{Z_{F2}} = \frac{\alpha_{F2} F_{F2}}{Z_{F2}} \]

\[ = f_{F2} \left( \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{H2}}{Z_{F2}}, \frac{Z_{H1}}{Z_{F1}}, N_{F2} \right) - \alpha_{F2} \theta_{F2} Z_{F2} \frac{Z_{F2}}{Z_{H1}} \]

\[ = g_{F2} \]

(134)
6.3 Trade Balance Condition

The trade balance condition that the value of Home’s exports equals the value of its imports:

\[ N_{H1}G_H^E, P_{G_H} = N_{F2}G_F^l, P_{G_F} \]

We can use the implications of Cobb-Douglas production and competitive markets to rewrite this condition in terms of fractions of product paid to factors of production. We have

\[ N_{H1}G_H^E, P_{G_H} = P_Y^e \lambda Y_F \]
\[ N_{F2}G_F^l, P_{G_F} = Y_H(1 - \epsilon) \lambda \frac{L_{H2}F}{L_{H2}} \]

so the trade balance condition can be written as

\[ P_Y^e \lambda Y_F = Y_H(1 - \epsilon) \lambda \frac{L_{H2}F}{L_{H2}} \] (135)

We have

\[ P_Y^e = \frac{A_{H2}/Z_{H2}^{(\delta + \gamma)/(1 - \lambda)}}{A_{F2}/Z_{F2}^{(\delta + \gamma)/(1 - \lambda)}} \]

from 112. Following similar steps as in the derivation of 102, we get

\[ Y_T^{Trade} = \kappa' \left( \left( \frac{Z_{H2}^{(\delta + \gamma)}}{P_{G_H}} \right) \left( \frac{Z_{H2}^{1 - (\delta + \gamma)}(1 - \epsilon)L_H}{1 - \epsilon} \right) \right)^{\epsilon} \left[ \left( \frac{Z_{F2}^{(\delta + \gamma)}}{P_{F2}} \right) \left( Z_{H2}^{1 - (\delta + \gamma)}((1 - \epsilon)L_H)^{1 - \epsilon} \right) \right] \] (136)

and

\[ Y_T^{Trade} = \kappa' \left( \left( \frac{Z_{H2}^{(\delta + \gamma)}}{P_{G_H}} \right) \left( \frac{Z_{H2}^{1 - (\delta + \gamma)}(1 - \epsilon)L_F}{1 - \epsilon} \right) \right)^{\epsilon} \left[ \left( \frac{Z_{F2}^{(\delta + \gamma)}}{P_{F2}} \right) \left( Z_{H2}^{1 - (\delta + \gamma)}((1 - \epsilon)L_F)^{1 - \epsilon} \right) \right] \] (137)

where

\[ \kappa' = \lambda^{\frac{\lambda}{\lambda - 1}}(1 - \epsilon)^{\frac{\lambda - 1}{\lambda - 1}} \epsilon^{\frac{\lambda}{\lambda - 1}} \]

and

\[ \left( \frac{Z_{F2}^{(\delta + \gamma)}}{P_{F2}^{(\delta + \gamma)/(1 - \lambda)}} \right) = \left( \frac{Z_{H2}^{(\delta + \gamma)}}{P_{G_H}^{(\delta + \gamma)/(1 - \lambda)}} \right) \]

based on the trade pattern condition 112. Plug 136, 137 and

\[ P_Y^e = \frac{A_{H2}/Z_{H2}^{(\delta + \gamma)/(1 - \lambda)}}{A_{F2}/Z_{F2}^{(\delta + \gamma)/(1 - \lambda)}} \]

into 135 to get

\[ L_{H2F} = \frac{\epsilon}{1 - \epsilon} P_Y^e \frac{Y_F}{Y_H} L_{H2} \]
\[ = \frac{\epsilon}{1 - \epsilon} P_Y^e \frac{Y_F}{Y_H} \left( \frac{Z_{H2}^{1 - (\delta + \gamma)}}{Z_{F2}^{1 - (\delta + \gamma)}} \right) \left( \frac{L_F}{L_H} (1 - \epsilon)L_H \right) \]
\[ = \epsilon L_F P_Y^e \frac{1}{A_{F2}} \left( \frac{Y_F}{Y_H} \right) \left( \frac{Z_{H2}^{1 - (\delta + \gamma)}}{Z_{F2}^{1 - (\delta + \gamma)}} \right)^{\epsilon} \]
\[ = \epsilon L_F \left( \frac{A_{H2}}{A_{F2}} \right)^{\frac{1}{\lambda - 1}} \left( \frac{Z_{H2}^{(\delta + \gamma)/(1 - \lambda)}}{Z_{F2}^{(\delta + \gamma)/(1 - \lambda)}} \right)^{\epsilon} \] (138)

We see that \( L_{H2F} \) is a function of \( Z_{F2}/Z_{H2} \) because \( \frac{Z_{H2}}{Z_{F2}} = Z_{H2}^{\eta} Z_{F2}^{1 - \eta} \), and \( L_{H2F} < L_{H2} = (1 - \epsilon)L_H \) because Home allocates labor to both \( G_H \) goods and \( G_F \) goods.
6.4 Level Effect

Compare Home’s level of final good output before and after trade at the moment that trade opens. Following similar steps as in section (5.4), we get the level of output before trade:

\[ Y_{H}^{Autarky} = \kappa \left[ \left( \frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}} \right) (Z_{H2}^{1-(\delta+\gamma)}) \left( \epsilon L_{H} \right) \right]^{\epsilon} \left[ \left( \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}} \right) Z_{H1}^{1-(\delta+\gamma)} \left( \left( 1 - \epsilon \right) L_{H} \right) \right]^{1-\epsilon} \]

and equation 136 we get the level after trade:

\[ Y_{H}^{Trade} = \kappa \left[ \left( \frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}} \right) (Z_{H2}^{1-(\delta+\gamma)}) \left( \epsilon L_{H} \right) \right]^{\epsilon} \left[ \left( \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}} \right) Z_{H1}^{1-(\delta+\gamma)} \left( \left( 1 - \epsilon \right) L_{H} \right) \right]^{1-\epsilon} \]

where \( \kappa' = \lambda \frac{\lambda}{\lambda-\gamma} \left( 1 - \epsilon \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \). The second term in the two expressions has the same value because, by assumption,

\[ \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}} = \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}} \]

The only difference between the two income levels is the spillover term. Under autarky, that term depends only on the quality \( Z_{H2} \) of the domestic good, whereas with trade it depends on the qualities of both the domestic and foreign goods: \( \tilde{Z}_{H2} = Z_{H2}^{n} Z_{F2}^{1-n} \). Thus the necessary and sufficient condition for trade to increase level output at the moment of trade, i.e. \( Y_{H}^{Trade} > Y_{H}^{Autarky} \) is \( Z_{F2} > Z_{H2} \), namely, that the externality to processed goods industry 1 derived from imports of type-2 intermediate goods exceeds the externality derived from domestically produced intermediate goods.

Foreign’s final good output before and after trade is

\[ Y_{F}^{Autarky} = \kappa \left[ \left( \frac{Z_{F1}^{(\delta+\gamma)}}{P_{G_{F1}}} \right) Z_{F2}^{1-(\delta+\gamma)} \left( \epsilon L_{F} \right) \right]^{\epsilon} \left[ \left( \frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}} \right) Z_{F1}^{1-(\delta+\gamma)} \left( \left( 1 - \epsilon \right) L_{F} \right) \right]^{1-\epsilon} \]

and

\[ Y_{F}^{Trade} = \kappa \left[ \left( \frac{Z_{F1}^{(\delta+\gamma)}}{P_{G_{F1}}} \right) Z_{F2}^{1-(\delta+\gamma)} \left( \epsilon L_{F} \right) \right]^{\epsilon} \left[ \left( \frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}} \right) Z_{F1}^{1-(\delta+\gamma)} \left( \left( 1 - \epsilon \right) L_{F} \right) \right]^{1-\epsilon} \]

Foreign is completely specialized, so the level effects at the moment when trade opens are exactly as in section (5.4).

6.5 Balanced Growth Rate Effect

Rewrite equations 136 and 137 as

\[ Y_{H}^{Trade} = \tilde{\kappa}_{H} Z_{H1}^{\gamma} Z_{H2}^{1-(\delta+\gamma)} (\epsilon + (\delta+\gamma)(1-\epsilon)) Z_{F2}^{(1-\gamma)(1-(\delta+\gamma))} \epsilon \]

and

\[ Y_{F}^{Trade} = \tilde{\kappa}_{F} Z_{H1}^{\gamma} Z_{H2}^{1-(\delta+\gamma)} (\epsilon + (\delta+\gamma)) Z_{F2}^{(1-\gamma)(1-(\delta+\gamma))} \epsilon \]

where the constants \( \tilde{\kappa}_{H} \) and \( \tilde{\kappa}_{F} \) are

\[ \tilde{\kappa}_{H} = \lambda \frac{\lambda}{\lambda-\gamma} \left( 1 - \epsilon \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \left( 1 - \epsilon \right)^{1-\epsilon} L_{H} \]

\[ = \lambda \frac{\lambda}{\lambda-\gamma} \left( 1 - \epsilon \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \frac{\lambda}{\lambda-\gamma} \left( A_{H1} \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \left( 1 - \epsilon \right)^{1-\epsilon} L_{H} \]

\[ \tilde{\kappa}_{F} = \lambda \frac{\lambda}{\lambda-\gamma} \left( 1 - \epsilon \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \frac{\lambda}{\lambda-\gamma} \left( A_{H2} \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \left( 1 - \epsilon \right)^{1-\epsilon} L_{F} \]

\[ = \lambda \frac{\lambda}{\lambda-\gamma} \left( 1 - \epsilon \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \frac{\lambda}{\lambda-\gamma} \left( A_{H2} \right) \frac{\lambda(1-\epsilon)}{\lambda-\gamma} e^{\epsilon} \left( 1 - \epsilon \right)^{1-\epsilon} L_{F} \]
Then the growth rates of $Y^{Trade}_H$ and $Y^{Trade}_F$ are

$$\frac{Y^{Trade}_H}{Y^{Trade}_H} = \Gamma \frac{Z^2}{Z^2} + \{\eta[1 - (\delta + \gamma)\epsilon + (\delta + \gamma)(1 - \epsilon)]\frac{Z^2}{Z^2} + \{(1 - \eta)[1 - (\delta + \gamma)\epsilon + (\delta + \gamma)(1 - \epsilon)]\frac{Z^2}{Z^2} \tag{139}\]

and

$$\frac{Y^{Trade}_F}{Y^{Trade}_F} = \Gamma \frac{Z^2}{Z^2} - (\delta + \gamma)\epsilon \frac{Z^2}{Z^2} + \{(1 - \eta)[1 - (\delta + \gamma)\epsilon + (\delta + \gamma)(1 - \epsilon)]\frac{Z^2}{Z^2} \tag{140}\]

where $\Gamma = 2(\delta + \gamma)\epsilon + (\delta + \gamma)$, and $\Gamma + (\delta + \gamma)\epsilon + (\delta + \gamma) = 1$. On the BGP, recall from section (3) that $c_i/Y_i$ is constant, so

$$\frac{Y^{Trade}_H}{Y^{Trade}_H} = \frac{Y^{Trade}_F}{Y^{Trade}_F} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} \tag{141}$$

By the same reasoning as before, balanced growth implies

$$g_H = g_H = g_F \tag{142}$$

Equate the right sides of 142 and 143, we get

$$\frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} = \frac{Z^2}{Z^2} \tag{143}$$

Substitute that into 141 and equate the right sides of 141 and 143 to get

$$\frac{\alpha F^2 Z^2}{Z^2} = \frac{\alpha H^2 Z^2}{Z^2} = \frac{\alpha H^2 Z^2}{Z^2} = \frac{\alpha H^2 Z^2}{Z^2} \tag{144}$$

Substitute this expression into 145 to get

$$\frac{Z^2}{Z^2} = \frac{\alpha F^2 Z^2}{Z^2} = \frac{\alpha H^2 Z^2}{Z^2} \tag{145}$$

Substitute 146 and 147 into 141 to get the balanced growth rate under incomplete specialization:

$$g^{Trade} = \frac{\delta}{1 - \delta} \sqrt{\frac{\alpha H^2 Z^2}{Z^2}}(\alpha H^2 Z^2)^{1 - \eta} - \frac{\rho}{1 - \delta} \tag{148}$$
We can compare this growth rate to the rates that prevail under autarky:

\[
g_i^{\text{Autarky}} = \frac{\delta}{1-\delta} \sqrt{\alpha_{i1} \theta_{i1} \alpha_{i2} \theta_{i2}} - \frac{\rho}{1-\delta}
\]

where \( i = H, F \).

Home’s growth rate increases if \( \alpha_F \theta_F > \alpha_H \theta_H \). Foreign’s growth rate increase if \( \alpha_{H1} \theta_{H1} (\alpha_H \theta_H)^{\eta} > \alpha_{F1} \theta_{F1} (\alpha_F \theta_F)^{\eta} \).

Finally, we have

\[
\left( \frac{Z_{H2}}{Z_{F2}} \right)^{\gamma} = \left( \frac{Z_{H2}}{Z_{H1}} \right)^{\gamma} \left( \frac{Z_{H1}}{Z_{F2}} \right)^{\gamma}
\]

Substitute this value into 128, 130 and 133. Then noting that all rates of return satisfy \( r = g^{\gamma} + \rho \) on the BGP, we can solve for \( N_{H1}^{*}, N_{H2}^{*} \) and \( N_{F2}^{*} \).

### 6.6 Transition Dynamics

Recall that from 128, 129, 130, 131, 133 and 134, we have the following results:

\[
g_{H1} = \frac{r_{H1}}{\delta} - \alpha_{H1} \theta_{H1} \frac{Z_{H1}}{Z_{H2}}
\]

(149)

where \( Z_{H2} = Z_{H2}^{1-\eta} \).

\[
g_{H2} = \frac{r_{H2}}{\delta} - \alpha_{H2} \theta_{H2} \frac{Z_{H2}}{Z_{H1}}
\]

(150)

\[
g_{F2} = \frac{r_{F2}}{\delta} - \alpha_{F2} \theta_{F2} \frac{Z_{F2}}{Z_{H1}}
\]

(151)

Within Home, the no-arbitrage condition requires that \( r_{H1} = r_{H2} \), as in the closed economy. We get the relation between \( r_{H1} \) and \( r_{F2} \) in a similar way to we did under complete specialization. Using the growth rates of final goods in both countries, 139 and 140, we get the difference between them:

\[
\frac{\dot{Y}_H^{\text{Trade}}}{Y_H^{\text{Trade}}} - \frac{\dot{Y}_F^{\text{Trade}}}{Y_F^{\text{Trade}}} = \{\eta [1 - (\delta + \gamma)] \epsilon + (\delta + \gamma) (1 - \epsilon) + (\delta + \gamma) \epsilon \} \frac{Z_{H2}}{Z_{H2}}
\]

\[
+ \{(1 - \eta) [1 - (\delta + \gamma)] \epsilon - [1 - (\delta + \gamma)] \epsilon - (\delta + \gamma) \} \frac{Z_{F2}}{Z_{F2}}
\]

(152)

\[
= \{\eta \epsilon - \eta (\delta + \gamma) \epsilon + (\delta + \gamma) \} \left( \frac{Z_{H2}}{Z_{H2}} - \frac{Z_{F2}}{Z_{F2}} \right)
\]

where \( \{\eta \epsilon - \eta (\delta + \gamma) \epsilon + (\delta + \gamma) \} = \eta [1 - (\delta + \gamma)] + (\delta + \gamma) > 0 \) given that \( 0 < (\eta + \gamma) < 1 \).

Recall that in equilibrium, by the Euler equation 54,

\[
\frac{\dot{Y}_H^{\text{Trade}}}{Y_H^{\text{Trade}}} = \frac{\dot{C}_H}{C_H} = r_{H1} - \rho = r_{H2} - \rho
\]

and

\[
\frac{\dot{Y}_F^{\text{Trade}}}{Y_F^{\text{Trade}}} = \frac{\dot{C}_F}{C_F} = r_{F2} - \rho
\]

By 152,

\[
r_{H1} - r_{F2} = \{\eta \epsilon - \eta (\delta + \gamma) \epsilon + (\delta + \gamma) \} \left( \frac{Z_{H2}}{Z_{H2}} - \frac{Z_{F2}}{Z_{F2}} \right)
\]

(153)
Combine $r_{H1} = r_{H2}$ with 149 and 150 to get

\[
\frac{(Z_{H1}/Z_{H2})}{Z_{H1}/Z_{H2}} = \frac{g_{H1} - g_{H2}}{Z_{H1}/Z_{H2}} = -\alpha_{H1}\theta_{H1}\left(\frac{Z_{H1}}{Z_{H2}}\right)^\eta\left(\frac{Z_{H1}}{Z_{F2}}\right)^{1-\eta} + \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}}
\] (154)

Now combine 153 with 150 and 151 to get

\[
\frac{(Z_{H2}/Z_{F2})}{Z_{H2}/Z_{F2}} = \frac{g_{H2} - g_{F2}}{Z_{H2}/Z_{F2}} = \frac{r_{H2} - r_{F2}}{\delta} - \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}
\]

\[
= \frac{\{\eta\epsilon - \eta(\delta + \gamma)\epsilon + (\delta + \gamma)\} (Z_{H2}/Z_{F2})}{\delta}
\]

\[
-\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}
\]

Which can be rearranged as

\[
-\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} = \frac{1 - \{\eta\epsilon - \eta(\delta + \gamma)\epsilon + (\delta + \gamma)\}}{\delta} \left(\frac{(Z_{H2}/Z_{F2})}{Z_{H2}/Z_{F2}}\right)
\]

\[
= \{\eta\epsilon + \eta(\delta + \gamma)\epsilon - \gamma\} (Z_{H2}/Z_{F2})
\]

\[
\Rightarrow \frac{(Z_{H2}/Z_{F2})}{Z_{H2}/Z_{F2}} = \frac{\delta}{-\eta\epsilon(1 - \delta - \gamma) - \gamma} \left[-\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}\right]
\] (155)

Define $u \equiv Z_{H1}/Z_{H2}$, $v \equiv Z_{H1}/Z_{F2}$, and $w \equiv Z_{H2}/Z_{F2}$. Note that $v = u \cdot w$. Rewrite 154 and 155 as

\[
\frac{\dot{u}}{u} = -\alpha_{H1}\theta_{H1}u^\eta v^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1}
\]

\[
= -\alpha_{H1}\theta_{H1}u^\eta (uw)^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1}
\]

\[
= -\alpha_{H1}\theta_{H1}u^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1}
\]

and

\[
\frac{\dot{w}}{w} = \frac{\delta}{-\eta\epsilon(1 - \delta - \gamma) - \gamma} \left[-\alpha_{H2}\theta_{H2}u^{-1} + \alpha_{F2}\theta_{F2}v^{-1}\right]
\]

\[
= \frac{\delta}{-\eta\epsilon(1 - \delta - \gamma) - \gamma} \left[\alpha_{H2}\theta_{H2}u^{-1} - \alpha_{F2}\theta_{F2}(uw)^{-1}\right]
\]

These equations yield

\[
\dot{u} = -\alpha_{H1}\theta_{H1}u^2w^{1-\eta} + \alpha_{H2}\theta_{H2}
\] (156)

and

\[
\dot{w} = \frac{\delta}{-\eta\epsilon(1 - \delta - \gamma) + \gamma} \left[\alpha_{H2}\theta_{H2}\frac{w}{u} - \alpha_{F2}\theta_{F2}\frac{1}{u}\right]
\] (157)

as the differential equations for $u$ and $w$ that only depend on $u$ and $w$. They are non-linear, so we linearize them by a Taylor expansion around steady state $u^*$ and $w^*$. From section (6.5), the values of the steady states are

\[
u^* \equiv \left(\frac{Z_{H1}}{Z_{H2}}\right)^* = \left(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}}\right)^{-1} \sqrt[\eta]{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}}
\] (158)
\[ w^* = \frac{Z_{H2}}{Z_{F2}} \]

The Taylor expansion for 156 is

\[ \dot{u} = -2\alpha_{H1}\theta_{H1}u^*(w^*)^{1-\eta}(u - u^*) - \alpha_{H1}\theta_{H1}(1 - \eta)(u^*)^2(w^*)^{-\eta}(w - w^*) \]  

Thus the locus of \( \dot{u} = 0 \) is

\[ u = \left( -\frac{1 - \eta}{2} \frac{u^*}{w^*} \right)w + (3 - \eta)u^* \]  

The Taylor expansion for 157 is

\[ \dot{w} = \frac{\delta}{\eta(1 - \delta - \gamma) + \gamma} \frac{\alpha_{H2}\theta_{H2}}{u^*} \frac{w - w^*}{u^*} + \frac{\delta}{\eta(1 - \delta - \gamma) + \gamma} \left( \frac{\alpha_{H2}\theta_{H2}}{w^*} \right) \]

where the second line follows from \( w^* = \frac{Z_{H2}}{Z_{F2}} \) = \( \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}} \), so \( \alpha_{H2}\theta_{H2}w^* - \alpha_{F2}\theta_{F2} = 0 \). The equilibrium locus \( \dot{w} = 0 \) is

\[ w = w^* \]  

Figure 1: w-u phase diagram

6.7 Stability of Incomplete Specialization

Recall that the trade pattern condition under incomplete specialization is

\[ \frac{(1 - \epsilon)L_H}{\epsilon L_F} \frac{(\delta + \gamma)(1 - \lambda)}{\delta} \geq \frac{A_{H2}/Z_{H2}^{(\delta + \gamma)(1 - \lambda)}}{A_{F2}/Z_{F2}^{(\delta + \gamma)(1 - \lambda)}} = P_Y \frac{A_{H1}/Z_{H1}^{\delta \lambda}}{A_{F1}/Z_{F1}^{\delta \lambda}} \]
and \( w = Z_{H2}/Z_{F2} \). Thus under incomplete specialization

\[
w \geq \left\{ \frac{(1-\gamma)H_2}{\alpha_{H2}} \right\}^{\gamma/\gamma(1-\gamma)}
\]

We have the following results:

1. If \( w = w^* \), then \( u \) converges to \( u^* \), and the world economy converges to BGP and stay on incomplete specialization.
2. If \( w < w^* \), then \( \dot{w} < 0 \). After certain time, the magnitude of \( w \) violates 166, and the whole economy evolves into complete specialization. And the complete specialization is stable as we proofed.
3. If \( w > w^* \), then \( \dot{w} > 0 \). World economy stays on incomplete specialization, but never converges to BGP. In this case, the difference of the growth rates of two countries converges to the constant

\[
(\delta + \frac{\delta^2}{\eta\epsilon(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2} \frac{1}{w^*}
\]

where

\[
u^* = \left( \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}} \right)^{-1} \sqrt{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}} \left( \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}} \right)^\gamma}
\]

To see this last result, note that from 152 and 157

\[
\frac{Y_{H}^{Trade}}{Y_{F}^{Trade}} = \frac{\{\eta - \eta(\delta + \gamma)\epsilon + (\delta + \gamma)\}Z_{H2}^{'}}{Z_{H2}^{'}} - \frac{Z_{F2}^{'}}{Z_{F2}^{'}}
\]

\[
= \frac{\{\eta - \eta(\delta + \gamma)\epsilon + (\delta + \gamma)\}}{\eta(1-\delta-\gamma)+\gamma}\frac{\delta}{\alpha_{H2}\theta_{H2}} \frac{w-w^*}{w} \frac{1}{w^*}
\]

\[
= (\delta + \frac{\delta^2}{\eta(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2} \frac{1}{w^*}
\]

\[
-\frac{(\delta + \frac{\delta^2}{\eta(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2} w^*}{w}
\]

\[
\Rightarrow (\delta + \frac{\delta^2}{\eta(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2} \frac{1}{w^*} \text{ as } w \to \infty
\]

6.8 Comparison with the Classical Ricardian Model

We can compare our comparative advantage result with that in the classical Ricardian model, explained by Dornbush, Fisher and Samuelson (1977), denoted hereafter as DFS (1977). We see several differences.

First, the classical Ricardian model does not have the externality discussed in section (5.4), so trade always has a positive effect on output level in the classical Ricardian model. In contrast, in our model the externality makes the level effect uncertain, as explained in section (5.4).

Second, DFS (1977), and most other discussion of Ricardian models, ignore the possibility of incomplete specialization. DFS (1977)’s definition of comparative advantage is similar to ours. DFS (1977) use a multiple good model, so for comparison with our model, we can set the number of goods to 2. Labor is the only factor used to produce trading goods. The wages in Home and Foreign are \( w_H \)
The unit labor requirements of good 1 in Home and Foreign are $A_{H1}$ and $A_{F1}$, respectively. The requirements for good 2 are $A_{H2}$ for Home and $A_{F2}$ for Foreign. Home produces the good with lower domestic labor cost. DFS assume perfect competition, so price equals marginal cost. Home produces good 1 if

$$w_H A_{H1} < w_F A_{F1}$$

which is equivalent to equation (2) in DFS(1977). Trade requires

$$w_H A_{H1} < w_F A_{F1} \quad \text{and} \quad w_H A_{H2} > w_F A_{F2}$$

or, of course, the opposite, but here we focus on the case in 168. That condition can be rewritten as

$$\frac{w_H}{w_F} < \frac{A_{F1}}{A_{H1}} \quad \text{and} \quad \frac{A_{F2}}{A_{H2}} < \frac{w_H}{w_F}$$

The two parts can be combined as

$$\frac{A_{F2}}{A_{H2}} < \frac{w_H}{w_F} < \frac{A_{F1}}{A_{H1}}$$

This last pair of inequalities means that the factor price ratio must be in the interior of the interval of unit-cost ratios. DFS(1977) do not discuss the possibility of corner solutions, where the factor price ratio is on the boundary of the unit-cost ratio interval. Note that 169 implies the comparative advantage expression

$$\frac{A_{H1}}{A_{H2}} < \frac{A_{F1}}{A_{F}}$$

Trade balance requires

$$\frac{w_H}{w_F} = \text{func}\left(\frac{L_H}{L_F}\right)$$

which is equation (10") and (11) in DFS (1977). Combining this condition with 169, we see that complete specialization requires that the population ratio be inside the interval of the ratios of unit costs:

$$\frac{A_{F2}}{A_{H2}} < \text{func}\left(\frac{L_H}{L_F}\right) < \frac{A_{F1}}{A_{H1}}$$

If the population ratio is outside the interval specified in 170, one economy will be incompletely specialized.

Third, in the classical Ricardian model, condition 170 is decided by endowment only. If, for example,

$$\text{func}\left(\frac{L_H}{L_F}\right) < \frac{A_{F2}}{A_{H2}}$$

we have an incompletely specialized world economy, and the world stays that way forever because $L_H$ and $L_F$ are given. The situation is quite different in our model, in which the economy can start in incomplete specialization and evolve into complete specialization.

Our model gives similar results depending on similar conditions, but our trading goods are produced under monopolistic competition rather than perfect competition, our prices are quality-adjusted, our factors of production for the trading goods are endogenous rather than endowed, our model is dynamic, and we also analyze the corner solution of incomplete specialization. In our model, Home chooses the good with lower quality-adjusted price, so the condition for trade is

$$\frac{P_{GH1}}{Z_{H1}^{\lambda}} \leq \frac{P_{GF1}}{Z_{F1}^{\lambda}} \quad \text{and} \quad \frac{P_{GH2}}{Z_{H2}^{\lambda}} \geq \frac{P_{GF2}}{Z_{F2}^{\lambda}}$$

$^5$DFS(1977) denote Home and Foreign wages as $w$ and $w^*$.

$^6$We change the notation again to be consistent with our model.
which is similar to the trade condition 168 in the Ricardian model. Complete specialization requires
\[
\frac{A_{H2}}{A_{F2}} \left( \frac{Z_{F2}}{Z_{H2}} \right)^{\lambda(1-\lambda)} > \frac{(1 - \epsilon)L_H}{\epsilon L_F} \left[ \frac{A_{H1}}{A_{F1}} \right]^{\lambda(1-\lambda)} \left( \frac{Z_{F1}}{Z_{H1}} \right)^{\lambda(1-\lambda)}
\]
which is similar to 170 in the Ricardian model. Comparative advantage here is given by the ratios of unit costs adjusted by qualities:
\[
\left( \frac{A_{H1}}{Z_{H1}^{\lambda(1-\lambda)}} \right) \left( \frac{A_{H2}}{Z_{H2}^{\lambda(1-\lambda)}} \right) \leq \left( \frac{A_{F1}}{Z_{F1}^{\lambda(1-\lambda)}} \right) \left( \frac{A_{F2}}{Z_{F2}^{\lambda(1-\lambda)}} \right)
\]