More Haste, Less Speed?
Signaling through Investment Timing

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June 22, 2016

Abstract

We consider a cash-constrained firm learning on the value of an irreversible project at a privately-known speed. Under perfect information, the optimal date of investment may be non-monotonic in the learning speed: better learning increases the value of experimenting further, but also the speed of updating. Under asymmetric information, the firm uses its investment timing to signal confidence in the project and raise cheaper capital from uninformed investors, which may generate timing distortions: investment is hurried when learning is sufficiently fast, and delayed otherwise. The severity of the cash constraint affects the magnitude of the distortion, but not its direction.

Keywords: Signaling, investment timing, financing of innovation, real options.

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1 Introduction

Innovation is a major driver of growth, and has accordingly been of central interest for scholars, policy-makers, and practitioners. In particular, much attention has been devoted to the question of the financing of innovation. An important concern is that financial frictions might lead to inefficient investment in R&D: the literature has indeed documented a wedge between the rate of return required by an entrepreneur to launch a R&D project and the rate of return of capital required by external investors to finance it (Hall and Lerner, 2010; Kerr and Nanda, 2014). One of the main reasons explaining such a wedge is asymmetric information: the innovator often has superior information than potential financiers about, e.g., his talent, the quality of his project, or the effort he puts in to run his business. Information and financial frictions are particularly relevant to innovative firms: first, they are often run by young entrepreneurs with no established reputation, so that information problems are more severe; second, a significant chunk of the innovation potential comes from small firms with little or no cash.

In this paper, we analyze how information and financial frictions impact entrepreneurs' investment timing policy by shaping their incentives to learn about their projects. Experimentation is actually a critical dimension of the innovation process: in the face of uncertainty, entrepreneurs first need to run tests so as to learn whether further investments are worthwhile. In a recent paper, Ewens *et al.* (2015) show that the recent fall in experimentation costs (cloud computing, accelerators...) has reduced financing constraints for projects with the greatest option value, and has mostly benefitted to young and inexperienced entrepreneurs. This suggests that information and financial frictions most hit small firms for which efficient learning is critical. In line with this idea, we examine how the timing of investment is distorted under asymmetric information about the quality of learning. This contrasts with the existing literature on investment timing under asymmetric information, which has instead focused on private information on

the initial value of the project, i.e., a variable orthogonal to learning. While there is a monotonic relationship between the optimal timing decision and the initial value of the project, the accuracy of learning has a non-monotonic impact on investment timing. This non-monotonicity in turn results in the timing distortion possibly going in both directions (hurried or delayed investment). Our paper therefore offers a simple and tractable model where information and financial frictions may generate either accelerated or deferred investment, and highlights the role of the learning environment as the main determinant of the direction of the timing distortion. This provides a novel interpretation of differences in timing or investment decisions across firms or markets: some firms may hurry investment while others delay investment because they have access to testing technologies with different precisions.

Specifically, we consider a continuous time model which combines the following ingredients: (a) an entrepreneur owns an irreversible project over which he learns as long as the project has not been launched (the entrepreneur holds a real option); (b) the precision of the signal he observes, hence the value of the option, is private information; (c) the entrepreneur is cash-constrained and needs outside funding to finance the project. Private information on the signal's precision implies that the entrepreneur has superior information about the value of the project compared to investors, hence a signaling problem: the date at which the entrepreneur launches the project conveys information on how confident he is, and the entrepreneur would have investors believe that he is as confident as possible in order to obtain cheaper credit.

We first analyze the benchmark case with complete information, and examine how the optimal timing of investment varies with the precision of the signal. The impact of a higher precision is two-fold: on the one hand, a more precise signal increases the value of the option, hence incentives to experiment further; on the other hand, it increases the speed of learning, so the entrepreneur becomes optimistic on the project at a faster pace,

raising incentives to invest.[1] When the prior net present value of the project is positive, the optimal investment date is single-peaked in the precision: it is optimal to invest early both when the signal is very precise (fast learning), and when it is very imprecise (there is little to be learnt from waiting). Put differently, the single-crossing property does not hold, which suggests that the information content of the investment timing is intrinsically ambiguous: if the entrepreneur wants to signal that he observes precise signals, it is a priori unclear whether he should invest early or late.

When the precision of the signal is private information, we first establish that the entrepreneur can always reach his complete information payoff whenever he holds sufficient cash: when a high share of the investment is internally financed, an entrepreneur with an imprecise signal is unwilling to distort his investment timing to pretend his signal is precise, as the cost of an inefficient investment policy dwarfs the benefit of cheaper capital. However, the investment timing has to be distorted whenever the entrepreneur holds insufficient cash, and the distortion is more severe the higher the cash shortage. This is because signaling concerns become more salient when a higher share of the project is financed with external funds. Interestingly, though, while the magnitude of the distortion depends on the severity of the cash constraint, the direction of the distortion is orthogonal to the entrepreneur's net worth.

Actually, the direction of the distortion only depends on the ranking of investment dates under complete information. Whenever an entrepreneur with a precise signal invests later (resp. earlier) than one with an imprecise signal under complete information, the former must defer (resp. accelerate) investment as compared to the efficient timing in order to prevent mimicking by the latter. The ranking of investment dates under complete information depends on two sets of parameters: (a) the initial value of the project, (b) the accuracy of learning. First, everything else equal, a higher prior value of the project makes

[1] Learning is modeled as bad news: either the entrepreneur learns that the project is bad for sure, or learns nothing, in which case he gets increasingly optimistic about the project as time passes.

early investment relatively more attractive to a slow-learning type whose option value is smaller, resulting in delayed investment by the fast-learning type. Second, everything else equal, fast-learning firms tend to prompt investment in an environment where learning is more accurate: indeed, the firm learning faster starts to lose its comparative learning advantage early when the slow-learning type learns sufficiently fast, as the latter quickly catches up on beliefs.

Empirically, the distortion in investment policy may have different interpretations. For instance, hurried investment as compared to the benchmark can be understood in a literal sense, but also implies that the probability of success conditional on investing is lower than in the benchmark (under-experimentation). Accordingly, the model allows predictions regarding timing decisions (e.g., the timing of move to subsequent phases of clinical trials, IPO timing, the timing of release of a new product or a new version of the same product, strategic acceleration or deferment of patent examination procedures...), but also on the probability of success conditional on investment. In either case, a major determinant of the direction of the distortion created by information and financial frictions is the precision of the learning technology (hence the speed of learning): in industries or markets characterized by fast learning, one should observe hurried investment. Hurried investment may for instance materialize in "grandstanding": Gompers (1996) has shown that young venture capital firms tend to take ventures public early because of signaling or reputation concerns; it may also materialize in under-experimentation, that is, an inefficiently high failure rate conditional on investment. Conversely, we should expect delayed investment, or over-experimentation in industries where learning is slower. This is consistent with evidence on the pharmaceutical industry, an industry where testing is extremely long.[2] For instance, Guedj and Scharfstein (2004) focus on drug development and establish that,

[2]In the market for drugs, DiMasi *et al.* (2003) estimate the time between the start of clinical testing and submission for approval to the FDA to 72.1 months. This takes into account neither the preclinical phase, which lasts between 1 and 6 years on average, nor the length of the FDA's approval process.

among less mature firms, i.e., those more likely to suffer from asymmetric information, those with low cash reserves tend to experiment longer than those with large reserves, and have a lower failure rate. This is also consistent with Henkel and Jell (2010), who establish that long patent examination deferments are particularly frequent in the pharmaceutical and chemical industries.

Our paper relates to the literature on real options (Dixit and Pindyck, 1994), and on exponential/Poisson learning, notably Keller *et al.* (2005) and Décamps and Mariotti (2004). It is also related to a strand of the recent literature on experimentation dealing with the impact of asymmetric information. Agency problems may involve adverse selection (private learning), moral hazard (unobservable learning effort), or both. Several papers have considered the question of the design of the optimal contract (or the optimal mechanism) for experimenting agents, for instance Manso (2011), Gerardi and Maestri (2012), or Halac *et al.* (2016b,a). In particular, a series of papers has focused on the question of the financing of experimentation (Bergemann and Hege, 1998, 2005; Hörner and Samuelson, 2013; Bouvard, 2014; Drugov and Macchiavello, 2014). A distinct class of papers have analyzed models with no commitment where the investment timing can be used as a signaling device (Grenadier and Malenko, 2011; Morellec and Schürhoff, 2011; Bustamante, 2012). Our paper, which also features a "real option signaling game", methodologically belongs to this stream of papers.

However, most of the aforementioned papers model private information on a variable which would affect the expected value of the project even in the absence of learning opportunities (e.g, the prior belief, or the cost of investment), and which has a monotonic effect the optimal timing decision. An exception is Halac *et al.* (2016b), who consider a setup with private information on the probability of success of the project, in which learning comes from observing past successes or failures, and derive a non-monotonicity result similar to ours under complete information. In all these papers, though, asym-

metric information generates timing distortions in one direction only. Instead, in our signaling game where private information is on the quality of learning, we derive not only a non-monotonic relationship between the optimal investment date and the precision of learning, but we also evidence that timing distortions may go both in the direction of hurried and delayed investment. In addition, the analysis provides an intuitive account of the determinants of the direction of the distortion, that is, the quality of the learning environment.

The paper is organized as follows: Section 2 is devoted to the presentation of the model. In Section 3, we characterize the optimal investment timing policy in the benchmark case of perfect information. In Section 4, we derive the unique equilibrium of the signaling game, and discuss the determinants of the direction of the timing distortion in Section 5. Section 6 reviews the empirical implications of the model. In Section 7, we discuss the robustness of the results. Finally, Section 8 concludes. All the proofs are relegated to the Appendix.

2 The model

A firm owns a project with *ex ante* uncertain value. With probability p_0, the project is of high quality, and yields a revenue $R > 0$. With probability $1 - p_0$, the project is of low quality, and yields zero revenue. Investment involves an irreversible cost $I \in (0, R)$. The firm is risk neutral and discounts future revenues and costs at rate $r > 0$. We typically have in mind a young entrepreneurship with no established reputation (or, more generally, an entrepreneur and a close set of well-informed stakeholders, e.g., initial investors, venture capitalist), but the firm could also be a more established firm facing cash constraints. For convenience, we call the firm "the entrepreneur" in what follows. When the entrepreneur is backed by a venture capitalist, we assume that the interests of insiders (the venture capitalist and the entrepreneur) are perfectly aligned, so that

they behave as a single entity.[3] Therefore, our view is complementary to the literature on venture capital which addresses contracting issues raised by misaligned incentives between venture capital firms and the venture they back (Gompers, 1996; Kaplan and Strömberg, 2003; Gompers and Lerner, 2004). Instead, we focus on information problems between the current shareholders/management (insiders) and the capital market (outsiders), and examine how they affect the timing of exit.

2.1 Learning environment

Time is continuous. The entrepreneur decides the date $t \geq 0$ at which he invests, if he ever does. The rationale behind waiting is that the entrepreneur learns about the quality of the project as long as investment has not taken place. We assume, following Décamps and Mariotti (2004), that he observes a signal modeled as a Poisson process with intensity $\lambda > 0$ if the project is of low quality, and with intensity 0 if the project is of high quality. Therefore, "no news is good news": a jump perfectly identifies a low-quality project, whereas the entrepreneur gets increasingly optimistic about project quality as long as nothing is observed. One may think of the pre-investment or learning period as a phase during which the entrepreneur runs tests (e.g., phase I of the FDA's drug review process), and of λ as the accuracy of the testing technology.[4]

The Poisson process is observed for free, so that the benefit of learning is only traded against the cost of delaying investment.[5] Let $s(\lambda, t)$ denote the total probability of observing no jump before date t :

$$s(\lambda, t) = p_0 + (1 - p_0)e^{-\lambda t}. \tag{1}$$

[3]Alternatively, the incentive problem between them is fully resolved by some optimal contract.

[4]Such an interpretation is consistent with the assumption that only bad news can be learnt. Indeed, the purpose of phase I investigations is to rule out major safety issues or important side-effects. Alternatively, in the software industry, experimentation may consist in designing an alpha/beta version in order to confidently reject the presence of security failures or bugs.

[5]One easily checks that the analysis of a model with costly learning would be qualitatively similar.

Using Bayes' rule, the entrepreneur's beliefs at date t conditional on no jump being observed read

$$p^*(\lambda, t) \equiv \frac{p_0}{s(\lambda, t)} = \frac{p_0}{p_0 + (1 - p_0)e^{-\lambda t}}. \tag{2}$$

We assume that the precision of the testing technology λ is private information to the entrepreneur. This captures the idea that experimentation requires some expertise. Actually, various testing procedures are usually available, and knowing which is the most efficient requires a specific talent. This is typically the case in the software industry: Tassey (2002) notably reports that "software testing is still more of an art than a science, and testing methods and resources are allocated based on the expert judgment of senior staff". This suggests that there is no such thing as a standardized testing procedure, and that picking the right procedure is a key strategic decision. In addition, outsiders are likely to observe neither the quality of the procedure chosen, nor the resources allocated to testing. For instance, in pharmaceuticals, a great deal of trials are subcontracted to contract research organizations (CROs), and this layer of delegation may raise additional information concerns (uncertainty about the quality of investigators, moral hazard). Asymmetric information on λ may accordingly capture in a reduced-form way the information problem created by such delegation. Finally, when the entrepreneur is backed by a venture capitalist, λ could also capture the privately-known ability of the venture capitalist to monitor the entrepreneur, or to contribute (e.g., through advising) to the growth of his venture.

2.2 The financing problem

The entrepreneur initially holds assets with value $A < I$. These assets can be liquid assets (cash or quasi-cash), or illiquid assets which can be pledged as collateral. In principle, these assets could continuously capitalize at any rate $r_0 \leq r$, but we focus here on the

case where $r_0 = 0$ for simplicity.[6] This allows to shut down the impact of time on the entrepreneur's cash position, and to focus on its impact on his beliefs only.

The entrepreneur needs to borrow cash at the date when he decides to invest in the project. Given that the payoff is either R or 0, the security that is issued (debt or equity) is irrelevant. We consider for simplicity that the entrepreneur raises cash from risk neutral and competitive investors by issuing equity, i.e., sells a fraction x of the firm in exchange for cash.

Importantly, we assume that investors can observe how long the entrepreneur has been learning before raising cash. For instance, pharmaceutical firms are required by the regulator (FDA, EMA) to disclose information regarding ongoing clinical trials.[7] Second, even when disclosure is not mandatory, it may be voluntary.[8] One can check that the equilibrium we derive in Section 4 would also obtain in the game where "date 0" is unobservable but can strategically be disclosed by the entrepreneur.[9] Alternatively, the market can observe the date of release of previous versions of a software, or the date at which the firm first filed a patent application. Finally, if the firm is backed by a venture capitalist, the date of the first round of financing is typically observable to investors.

However, whether a signal revealing a bad project is observed by all parties or privately observed by the entrepreneur is irrelevant here, as the entrepreneur never solicits funding if he knows the project to be bad. Indeed, the cash he must invest out of pocket (or the collateral) would then be lost for sure.[10]

[6] We discuss in Section 7.2 and show in the Appendix how our results extend to the general case $r_0 \leq r$.

[7] For instance, the FDAAA801 imposes that clinical trials be registered at most 21 days after the first patient is enrolled.

[8] Actually, several big pharmaceutical companies (e.g., Novartis, Sanofi, GSK) publicly commit on their websites to report information about their trials (from phase I to IV), including information for which disclosure is not mandatory.

[9] Such an equilibrium where the entrepreneur always discloses "date 0" is supported by out-of-equilibrium specifying that an entrepreneur that does not disclose holds the most conservative beliefs p_0.

[10] This assumes that the entrepreneur cannot take the money and run, i.e., raise cash and not invest in the project. Implicitly, we therefore assume that new shareholders can monitor the entrepreneur and impose investment at the date when capital is raised.

3 Complete information benchmark

Before we turn to the signaling problem raised by private information on λ, let us first examine how λ affects the entrepreneur's behavior in the benchmark case of perfect information. In this case, even if investors do not observe the entrepreneur's beliefs on the project, they can perfectly back them out using (2). Since investors are competitive, the entrepreneur obtains the full expected value of the project regardless of the date at which he solicits funding. Let $W^*(\lambda, t)$ denote the expected discounted payoff at date 0 of a entrepreneur learning at speed λ when he invests at date t (conditional on no bad news). One has:

$$\begin{aligned} W^*(\lambda, t) &= e^{-rt} s(\lambda, t) \left(p^*(\lambda, t) R - I\right) \\ &= e^{-rt} \left(p_0 R - I + (1 - s(\lambda, t)) I\right). \end{aligned}$$

The expression of W^* evidences the trade-off faced by the entrepreneur between discounting and learning: while waiting delays the realization of the payoff, it also allows to keep the option not to invest open. The value of this option depends on $1 - s(\lambda, t)$, the probability that bad news accrues by date t, in which case the entrepreneur does not invest and avoids sinking the outlay I. This probability increases in λ : better learning increases the value of the option.

The optimal investment date t^* reflects this tradeoff. It is such that:[11]

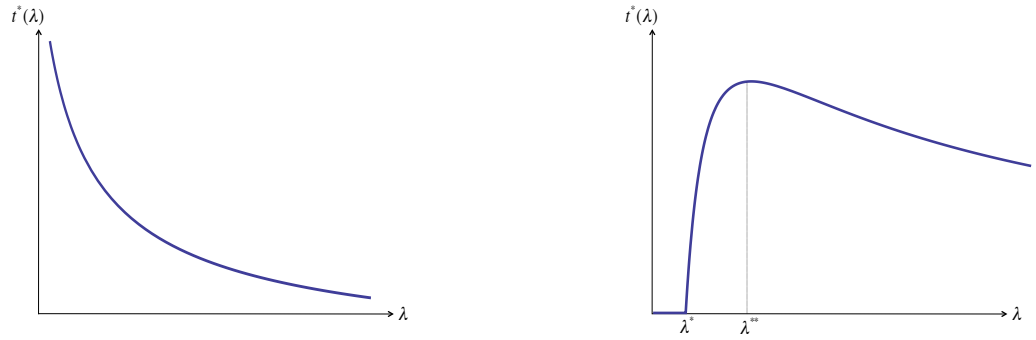
$$t^* = \underset{t \geq 0}{\operatorname{argmax}} W^*(\lambda, t) = \max\left(-\frac{1}{\lambda} \ln \frac{p_0 r (R - I)}{(1 - p_0)(\lambda + r) I}, 0\right). \tag{3}$$

It is clear t^* is nonincreasing in R, p_0 and r, and nondecreasing in I. The comparative statics of t^* with respect to λ is less immediate, as shown in the following Proposition.

[11] Notice that it is fine to solve this stopping problem by maximizing the date-0 expected payoff since the entrepreneur perfectly forecasts at date 0 his posterior beliefs at all future dates.

Proposition 1 *The impact of λ on the optimal investment date t^* depends on the prior NPV of the project:*

- *If $p_0R - I < 0$, $t^*(\lambda)$ is decreasing in λ;*
- *If $p_0R - I \geq 0$, there exist λ^* and λ^{**}, with $0 \leq \lambda^* < \lambda^{**}$, such that $t^*(\lambda) = 0$ for $\lambda \leq \lambda^*$, $t^*(\lambda)$ is increasing on $(\lambda^*, \lambda^{**})$ and decreasing on $(\lambda^{**}, +\infty)$.*



(a) $p_0R - I < 0$. (b) $p_0R - I \geq 0$.

Figure 1: Learning accuracy and the efficient timing of investment

The comparative statics of t^* in λ reflects the coexistence of two opposite effects: on the one hand, the entrepreneur triggers investment when his beliefs first reach a threshold $p_{t^*} = \frac{(\lambda+r)I}{rR+\lambda I}$. This term increases in λ, as an entrepreneur observing more accurate information is willing to wait until more optimistic before investing (the value of his option is higher); on the other hand, he also reaches a given threshold at a faster speed. When $p_0R - I < 0$, the latter effect dominates, but, when $p_0R - I \geq 0$, the two effects alternatively dominate, and the optimal investment date is single-peaked in λ.[12] Notice that such a non-monotonicity would obtain in more general settings as long as the optimal strategy of the entrepreneur is a trigger strategy whereby he invests as soon as the

[12]Halac *et al.* (2016b,a) derive an analogous non-monotonicity result in models with costly information acquisition. In their case, though, the case where t^* is decreasing in λ never obtains. Indeed, they focus on the case where costly experimentation is valuable, which is the "counterpart" of our positive NPV case. In the opposite case where experimentation is too costly, there is no learning, so the investment date does not reflect the learning speed. This non-monotonicity also arises in Bloch *et al.* (2011) in a duopoly model of entry timing.

expected value of the project (conditional on the past history) is above some cutoff. Such trigger strategies are optimal for a wide class of joint distribution of payoffs and signals when the action set is binary. Since our focus here is on investment timing rather than the intensity of investment, we consider a stopping game with only two actions, invest or continue learning, but it would be interesting to consider as well the intensive margin, and examine the impact of the cash and information frictions when investment is scalable.

This non-monotonicity makes the signaling content of investment timing intrinsically ambiguous: for instance, early investment could stem from an entrepreneur with a high λ who has learnt fast, or from an entrepreneur with a low λ whose option has little value. In other words, the single-crossing property does not hold. In the next section, we explore in detail how this non-monotonicity shapes signaling incentives under asymmetric information.

4 Incomplete information

We now assume λ to be private information: $\lambda \in \left\{\underline{\lambda}, \overline{\lambda}\right\}$, with $0 \leq \underline{\lambda} < \overline{\lambda}$, and $\Pr(\lambda = \overline{\lambda}) = q_0$. To simplify notation, let us denote $\overline{t}^* = t^*(\overline{\lambda})$ and $\underline{t}^* = t^*(\underline{\lambda})$.

4.1 Payoffs

The cost of outside capital reflects investors' beliefs over the quality of the project. Investors know how long the entrepreneur has been waiting, but do not know his true beliefs because of private information on λ. If q denotes the probability that the market assigns to the entrepreneur being of type $\overline{\lambda}$, the market perceives the probability of success at date t to be

$$p\left(q,t\right) = \frac{p_0}{qs\left(\overline{\lambda},t\right) + (1-q)s\left(\underline{\lambda},t\right)}. \tag{4}$$

For simplicity, let us consider the case where the assets A are liquid, so that the

entrepreneur borrows $I - A$.[13] Since investors are competitive, the entrepreneur needs to sell a share $x(q,t)$ of the firm such that

$$x(q,t)p\left(q,t\right)R = I - A. \tag{5}$$

There are two implicit assumptions behind (5): first, the entrepreneur does not borrow more than needed. This can be easily shown to be optimal. Indeed, the entrepreneur is risk neutral, so the only motive for selling shares is the cash shortage. For the good entrepreneur, i.e., the entrepreneur with the more precise signal, the cost of outside capital is no smaller than the cost of inside funds, as the market can never be more optimistic about the project than he is. Therefore, it is a weakly dominated strategy for him to sell more shares than needed, and the market should accordingly interpret superfluous issuance as coming from a bad entrepreneur.[14] Second, we assume that the entrepreneur does not underprice by selling his shares below their expected value to investors. Generically, equilibria involving money burning might exist, but they would feature timing distortions qualitatively similar to those we derive, and we ignore them here for simplicity.[15]

Let $W\left(\lambda,q,t\right)$ denote the expected discounted payoff at date 0 of type $\lambda \in \{\underline{\lambda},\overline{\lambda}\}$ when he invests at date t, and is perceived as type $\overline{\lambda}$ with probability q. We have

$$W\left(\lambda,q,t\right) = e^{-rt}s\left(\lambda,t\right)\left[p^{*}\left(\lambda,t\right)\left(1-x(q,t)\right)R - A\right]. \tag{6}$$

One remarks that $W(\overline{\lambda},1,t) = W^{*}(\overline{\lambda},t)$ and $W(\underline{\lambda},0,t) = W^{*}(\underline{\lambda},t)$, i.e., the entrepreneur's expected payoff under asymmetric information is the same as under perfect

[13] If A is interpreted as the liquidation value of assets, the entrepreneur has to borrow I, but financiers are then able to seize A in case of failure of the project, so the two interpretations are formally equivalent.

[14] This argument relies on a refinement like D1. See Section 4.2.1.

[15] Actually, since a distortion of investment timing involves a second-order loss around the efficient date, while leaving money on the table generates a first-order loss, timing distortions would still arise. In addition, the distortion would go in the same direction given that whether relaxing the incentive constraint implies hurrying or delaying investment does not depend on the amount of cash raised (see below, in Section 5.1). With two instruments at hand allowing to signal his type, an entrepreneur would simply use a combination of both instruments, so that timing distortions should be in magnitude smaller.

information as long as investors hold true beliefs on λ, a consequence of competition among investors. However, when the market holds wrong beliefs, the cost of internal and external funds differ, and there is a benefit (cost) for the entrepreneur from being mistakenly perceived as a good (bad) type: $W(\lambda, q, t)$ is nondecreasing in q.[16]

From (2) and (6), we derive:

$$W\left(\overline{\lambda}, q, t\right) - W\left(\underline{\lambda}, q, t\right) = Ae^{-rt} f(t), \tag{7}$$

where

$$f(t) \equiv s(\underline{\lambda}, t) - s(\overline{\lambda}, t) = (1 - p_0)(e^{-\underline{\lambda}t} - e^{-\overline{\lambda}t}). \tag{8}$$

For given investor beliefs and a given investment date, the only difference between two entrepreneurs with different signal precisions lies in the value of their options: the good type (rightly) abstains from investing with probability $1 - s(\overline{\lambda}, t)$, in which case he saves the outlay (Ae^{-rt} in present value), while the bad type does so with probability $1 - s(\underline{\lambda}, t)$ only. It is easy to see that $f(t)$ is nonnegative, as the good type is more likely to learn bad news, and that $f(0) = 0$, as both types differ only to the extent that some learning has taken place. Finally, f is single-peaked in t. This has to do with the property that learning exhibits decreasing returns with such Poisson learning: the probability of learning bad news before t increases, but the marginal increase is decreasing with time. This implies that the comparative learning dynamics is characterized by two phases: a first phase in which the good type learns more at the margin (as time goes by, the good type becomes increasingly more optimistic than the bad type), and a second phase in which the bad type catches up on beliefs: in the limit, there is perfect learning for any positive λ, so the difference between types becomes negligible.

[16] Remark also, using (5), that $W(\lambda, q, t) = e^{-rt} s(\lambda, t) \left[(1 - x(q, t))\left(p^*(\lambda, t)R - I\right) + x(q, t)\left(p(q, t)R - I\right)\right]$. Accordingly, one can interpret our problem as a problem of managerial myopia, where the entrepreneur cares about a weighted average of the true value of the firm and the stock price at the date of investment. Our model is therefore suited to study the impact of information frictions on timing decisions in a more general class of corporate governance problems.

Finally, let us denote by $\widetilde{f}(t) \equiv e^{-rt} f(t)$ the discounted difference in option values. $\widetilde{f}(t)$ is also single-peaked in t, and reaches its maximum at

$$t_0 \equiv \frac{\ln(\overline{\lambda} + r) - \ln(\underline{\lambda} + r)}{\overline{\lambda} - \underline{\lambda}} > 0.$$

4.2 Equilibrium analysis

Let us first notice that asymmetric information has no bite when $\overline{t}^* = 0$. In this case, $\underline{t}^* = 0$ as well, using (3). Therefore, both types can reach their complete information payoffs by investing at date 0, in which case investors' beliefs are irrelevant ($p(q, 0) = p_0$ for all q). In what follows, we therefore restrict attention to $\overline{t}^* > 0$.

4.2.1 Equilibrium definition and concept

We look for perfect Bayesian equilibria satisfying D1. Whenever D1 is not enough to guarantee uniqueness, we select the Pareto-dominant equilibrium, or least-cost equilibrium.[17] A pure-strategy equilibrium features investment dates $\underline{t}$ and $\overline{t}$ (conditional on no jump) for types $\underline{\lambda}$ and $\overline{\lambda}$, and a belief function $q(t)$, which assigns a probability that investment at date t comes from type $\overline{\lambda}$.

4.2.2 Separating equilibria

We first look for separating equilibria with $\underline{t} \neq \overline{t}$. First of all, it is standard that, in any separating equilibrium, one must have $\underline{t} = \underline{t}^*$. If $\underline{t} \neq \underline{t}^*$, $\underline{\lambda}$ could always increase his payoff by playing $\underline{t}^*$: even it is does not improve investors' beliefs, it yields a higher expected

[17] D1 imposes to attribute a deviation to some out-of-equilibrium date t to the type with the stronger incentive to deviate to t (see Section 9.9 for details on how D1 applies to our signaling game). We will see that, whenever there is equilibrium multiplicity, the only equilibria are separating, meaning that they can be Pareto-ranked.

payoff. A separating equilibrium $(\underline{t}^*, \bar{t})$ exists if and only if the following constraints hold:

$$\begin{cases} W(\overline{\lambda}, 1, \bar{t}) \geq W(\overline{\lambda}, 0, \underline{t}^*) & \text{(9a)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \bar{t}) & \text{(9b)} \\ W(\overline{\lambda}, 1, \bar{t}) \geq W(\overline{\lambda}, q(t), t) \qquad \text{for all } t \notin \left\{\underline{t}^*, \bar{t}\right\} & \text{(9c)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, q(t), t) \qquad \text{for all } t \notin \left\{\underline{t}^*, \bar{t}\right\} & \text{(9d)} \end{cases}$$

These incentive constraints specify that each type must prefer his equilibrium action to that of the other type, and has no profitable off-path deviation. Before providing conditions for a separating equilibrium, let us remark that D1 allows to restrict the set of possible equilibrium dates $\bar{t}$.

Lemma 1 *In any separating equilibrium satisfying D1, $\overline{\lambda}$ invests at a date comprised between $\bar{t}^*$ and t_0 : $\bar{t} \in [\min\left(t_0, \bar{t}^*\right), \max\left(t_0, \bar{t}^*\right)]$.*

Since the difference in option values between both types is maximum at $t = t_0$, a deviation to an off-equilibrium date between t_0 and $\bar{t}$ (i.e., "in the direction of t_0") is relatively more beneficial to the good type (for any out-of-equilibrium beliefs associated to this deviation), and D1 imposes to attribute such a deviation to $\overline{\lambda}$. Given this restriction, Lemma 1 has to hold. Otherwise, by deviating in the direction of t_0, the good type could get closer to his preferred action, while still being perceived as good.

The next Proposition provides necessary and sufficient conditions for the existence of a separating equilibrium.

Proposition 2 *There exists a separating equilibrium if and only if*

$$A \geq A_1 \equiv \max\left(0, \frac{W^*\left(\overline{\lambda}, t_0\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{\widetilde{f}(t_0)}\right).$$

In particular, $(\underline{t}^*, \overline{t}^*)$ *is an equilibrium if and only if*

$$A \geq A_0 \equiv \frac{W^*\left(\overline{\lambda}, \overline{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{\widetilde{f}(\overline{t}^*)} \in (A_1, I).$$

In a separating equilibrium, the sorting of types is achieved when the difference between each type's beliefs on the project is such that investing his cash A out of pocket at date $\overline{t}$ is too costly to the bad type. Whenever the cash constraint is sufficiently soft ($A \geq A_0$), the entrepreneur can reach his complete information payoff. Indeed, a high share of the investment is then externally financed, and the benefit of cheap capital does not compensate the loss due to the inefficiency of the investment policy, so that mimicking is not a concern. When $A \in (A_1, A_0)$, the timing must be distorted, but the entrepreneur still has enough cash to make separation possible. Notice that $A_0 > 0$, so efficiency is never attainable when A is too small. However, one may have $A_1 = 0$, in which case there exists a separating equilibrium despite the entrepreneur holding no cash. This is because the preferences of each type over investment timing is a sorting force *per se*: it is intrinsically costly for an entrepreneur of a given type to mimic the other type, even when the project is fully externally financed. Accordingly, their preferences may be sufficiently different for a separating equilibrium to exist even when the entrepreneur has no cash.

4.2.3 Pooling and semi-pooling equilibria

We now characterize pooling and semi-pooling equilibria:

Proposition 3 *If* $A \geq A_1$, *any equilibrium is fully separating.*

In addition, $\exists\, A_2 \leq A_1$ *such that:*

- *If* $A_2 \leq A < A_1$, *the unique equilibrium is such that* $\overline{t} = t_0$, *and* $\underline{\lambda}$ *randomizes between* t_0 *and* $\underline{t}^*$,
- *If* $A < A_2$, *the unique equilibrium is pooling:* $\underline{t} = \overline{t} = t_0$.

In the Appendix, we derive the cutoff value A_2 :

$$A_2 = \max\left(0, I - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0 \widetilde{f}(t_0)}\right).$$

Notice that the pooling equilibrium never exists if $A_2 = 0$, i.e., when $q_0 \leq \frac{W^*(\underline{\lambda},\underline{t}^*)-W^*(\underline{\lambda},t_0)}{I\widetilde{f}(t_0)}$. This corresponds to instances where the prior probability q_0 is small enough, in which case the bad type is unwilling to distort his investment strategy, as the gain of being perceived as the average type is too small.

Two findings emerge from Proposition 3. First, the equilibrium cannot involve any pooling when a separating equilibrium exists. Second, when there is pooling in equilibrium, it must be at date t_0. Investment at t_0 actually corresponds to maximal incentives, as it maximizes the difference between the expected cost for each type to wrongly invest in the project.

We conclude the equilibrium analysis by the following Corollary:

Corollary 1 *There is a unique least-cost equilibrium satisfying D1.*

The only possible range of multiplicity is $A \geq A_1$. In this case, there may be a continuum of separating equilibria, but pooling is then impossible. Therefore, all these equilibria can be Pareto-ranked, and there is consequently a unique least-cost separating equilibrium.

5 Timing distortions

In this Section, we examine how the equilibrium depends on the primitives of the model: the severity of the cash constraint, the precision of learning, and the prior value of the project. In particular, we characterize the situations where the information problem results in delayed investment, and those in which hurried equilibrium arises, which notably allows to derive empirical predictions.

In Lemma 1, we have shown that investment is hurried (resp. delayed) if $t_0 < \bar{t}^*$ (resp. $t_0 > \bar{t}^*$). Remarking that $W^*(\overline{\lambda}, t) = W^*(\underline{\lambda}, t) + I\widetilde{f}(t)$, one derives that $\bar{t}^* \leq t_0$ if and only if $\underline{t}^* \leq \bar{t}^*$. Therefore, the direction of the distortion is simply given by the ordering of the efficient investment dates. If the good type invests earlier (resp. later) than the bad type under complete information, then he should accelerate (resp. defer) investment as compared to the efficient timing to deter mimicking by the bad type.

5.1 The impact of the cash constraint

As neither $\bar{t}^*$ nor t_0 depends on A, the direction of the distortion does not depend on the severity of cash constraint. However, A does affect the magnitude of this distortion. When the cash constraint is soft, the complete information payoffs are attained. Otherwise, the good type needs to distort his investment date in the direction of t_0 in order to prevent mimicking by the bad type. When the information problem becomes too severe (A is too small), the only solution for the good type is to invest at t_0 to maximize the difference in option values. When this is not sufficient to achieve separation, the good type must be (fully or partially) pooled with the bad type. Proposition 4 establishes formally that the distortion incurred by the good type increases with the cash shortage.

Proposition 4 *The magnitude of the distortion $|\bar{t}^* - \bar{t}|$ is nonincreasing in A.*

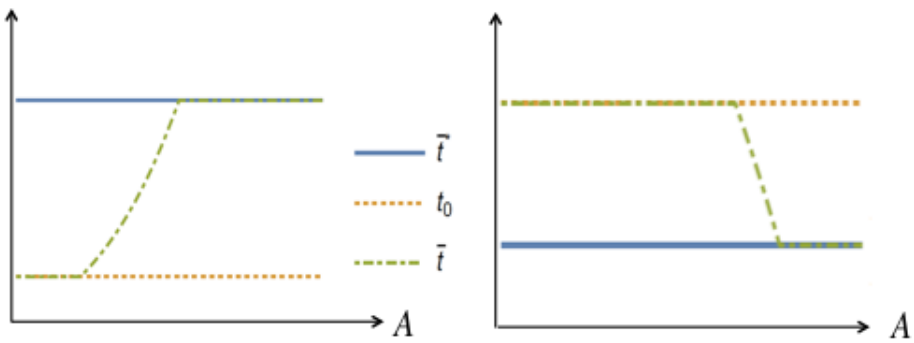


Figure 2: Cash constraint and distortion

Rewriting (9b) as $W(\underline{\lambda}, 1, t) \geq W^*(\overline{\lambda}, t) - A\widetilde{f}(t)$, one sees that the good type needs to distort his timing in the direction of t_0 to relax the incentive constraint of the bad type (see Lemma 1). Such a distortion has a stronger impact on the incentive constraint whenever A increases, and the distortion becomes accordingly smaller (in absolute value). A captures the stake of the entrepreneur in his project, i.e., the share of the investment which is financed internally. As the cash shortage problem improves (that is, as A increases), the benefit from fooling investors decreases, and the cost from distorting the timing away from one's preferred timing policy increases. By improving the sorting of types, a higher A therefore attenuates the information problem, and decreases the welfare loss.[18]

5.2 The impact of the learning speed

From Proposition 1, one always has $\overline{t}^* < \underline{t}^*$ when $p_0R - I < 0$, so that investment can only be hurried. When $p_0R - I \geq 0$, there are two cases:

- $\overline{\lambda} \leq \lambda^{**}$, in which case $\underline{t}^* < \overline{t}^*$ and investment can only be delayed,
- $\overline{\lambda} > \lambda^{**}$, in which case there exists a cutoff value $\underline{\lambda}_0$ such that $t^*(\underline{\lambda}_0) = t^*(\overline{\lambda})$. Investment is delayed if $\underline{\lambda} \leq \underline{\lambda}_0$, and hurried otherwise.

Accordingly, an empirical prediction is that investment should be hurried in markets where learning is fast (i.e., where both $\underline{\lambda}$ and $\overline{\lambda}$ are large enough), and delayed in markets where learning is slow (where $\underline{\lambda}$ is small enough). To understand this, recall that f (hence $\widetilde{f}$) reflects the comparative learning dynamics: in a first phase, the beliefs of both types diverge apart, with the good type learning faster than the bad type at the margin; in a second phase, the bad type catches up on beliefs. When both types learn sufficiently fast, the phase during which the good type learns faster stops early. In this case, signaling fast learning imposes to invest early to make sure that the bad type is sufficiently less

[18] More generally, one shows that the expected total welfare (i.e., including the welfare of $\underline{\lambda}$) is nondecreasing in A, but we omit the proof for concision.

confident than the good type about the project. Conversely, when the slow-learning type learns little enough, the first phase stops later, and the good type should exploit as much as possible his comparative learning advantage by waiting longer.

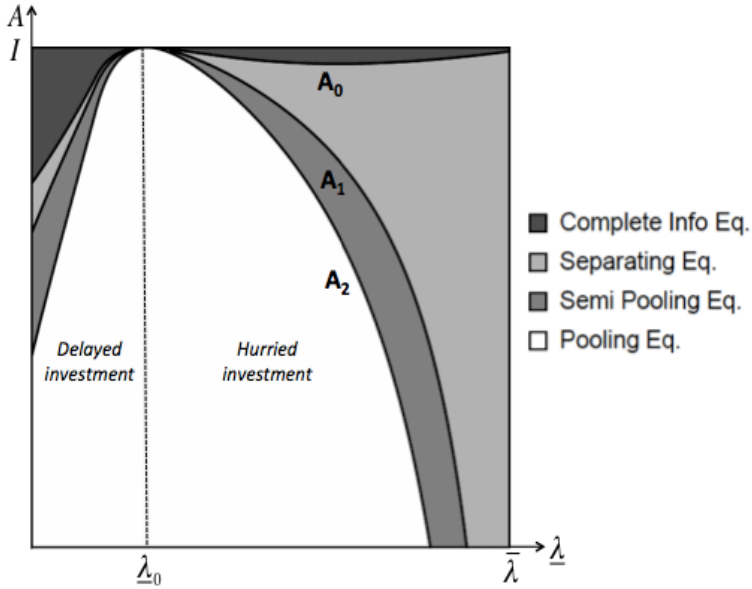


Figure 3: Equilibrium characterization

We summarize in Figure 3 the different equilibrium regions in the space $(\underline{\lambda}, A)$ in the positive NPV case. The figure displays that higher values of A make it more likely that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium, and, if not, that there exists a separating equilibrium. It also shows that separation is obtained by delaying investment when $\underline{\lambda}$ is sufficiently small, and by hurrying investment otherwise, regardless of the value of A.

Since λ affects both the value of option to learn and the speed of updating, it is a priori unclear whether an entrepreneur with a more precise signal is more or less confident upon investing. Indeed, one could compensate a lower accuracy of learning by waiting longer. Under complete information, this can never be the case: even when a fast-learning entrepreneur optimally invests sooner, he is always more confident at the date when he invests. Under asymmetric information, this is a fortiori true when investment is delayed. However, when investment is hurried, we show that the timing distortion may be large

enough to generate a reversal of beliefs, in that the bad type becomes more optimistic on the project upon investing than the good type:

Proposition 5 *Suppose that* $\overline{t}^* < \underline{t}^*$ *(investment is hurried).*

Then there exists a cutoff λ_1 *such that*

$$\underline{\lambda} > \lambda_1 \Rightarrow \exists \underline{A} \in (A_1, A_0) \text{ such that } p^*(\underline{\lambda}, \underline{t}^*) > p^*(\overline{\lambda}, \overline{t}) \text{ for } A_1 \leq A < \underline{A}.$$

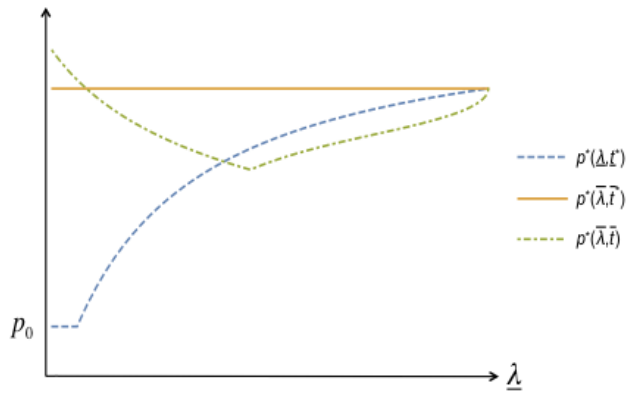


Figure 4: Belief reversal

5.3 The impact of the initial value of the project

Before closing this section, let us characterize the equilibrium as a function of p_0.[19]

Proposition 6 *There exist* $\{\underline{\underline{p}}, \underline{p}, \hat{p}, \overline{p}, \overline{\overline{p}}\}$ *with* $0 < \underline{\underline{p}} < \underline{p} < \hat{p} < \overline{p} < \overline{\overline{p}} < 1$ *such that:*

- *If* $p_0 \notin \left(\underline{\underline{p}}, \overline{\overline{p}}\right)$, *investment is efficient:* $\overline{t} = \overline{t}^*$;
- *If* $p_0 \in (\underline{\underline{p}}, \hat{p})$, *investment is hurried:* $\overline{t} < \overline{t}^*$; *the equilibrium is fully separating if* $p_0 \in (\underline{\underline{p}}, \underline{p})$, *it is semi-pooling or pooling* ($\overline{t} = t_0$) *if* $p_0 \in (\underline{p}, \hat{p})$;
- *If* $p_0 \in (\hat{p}, \overline{\overline{p}})$, *investment is delayed:* $\overline{t} > \overline{t}^*$; *the equilibrium is semi-pooling or pooling* ($\overline{t} = t_0$) *if* $p_0 \in (\hat{p}, \overline{p})$, *it is fully separating if* $p_0 \in (\overline{p}, \overline{\overline{p}})$.

[19] Notice that the equilibrium characterization as a function of R is entirely similar, but we omit it for concision.

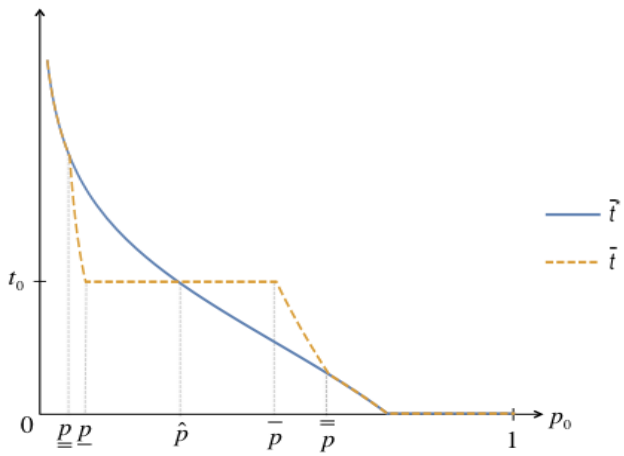


Figure 5: The equilibrium investment date $\bar{t}$ as a function of p_0

The investment timing is not distorted when the expected value of the project is either large or small enough. This is because there is then little uncertainty on the project, so that the cost of outside capital is not too different for both types under perfect information, which reduces mimicking concerns. In the intermediate region where investment has to be distorted, investment should be hurried when the expected value of the project is small, and delayed otherwise. Actually, a higher initial value increases the cost of delaying the project, relatively more so for the slow-learning type, who has a smaller option value, which induces the fast-learning type to delay investment.

Accordingly, an empirical prediction of the model is that projects with a low expected value (a low p_0 or R) are hurried when the cash constraint is too severe, while projects with better prospects are delayed. Notice also that, for any $A < I$, the equilibrium spans all the different regions (separating and pooling equilibrium, hurried and delayed investment) when p_0 varies from 0 to 1. Finally, one notices from Figure 5 that the magnitude of the distortion is non-monotone in p_0 ($|\bar{t} - \bar{t}^*|$ changes monotonicity at least twice).

6 Empirical predictions

A first prediction of the model is that information and financial constraints do affect investment: otherwise identical firms with different levels of cash have different investment

policies. This is consistent with the finding that cash-rich firms have an investment policy which is less sensitive to their net worth than cash-constrained firms (see Hubbard (1998) for a survey). More specifically, our model predicts that inefficient investment may take both the form of accelerated and deferred investment according to the expected value of the project and the shape of the learning curves in the market. First, projects with a high prior expected value tend to be delayed, while those with a low prior value tend to be hurried. This suggests that there should be too much and too risky investment in markets with poor prospects.[20] Conversely, there should be too little and insufficiently risky investment in markets with better prospects. Second, in industries with slow learning, cash-constrained firms should invest later than unconstrained firms, which implies that they should succeed with a higher probability conditional on investing. This is consistent with Guedj and Scharfstein (2004), who show that, among small drug companies, i.e., companies more likely to suffer from asymmetric information, those with low cash reserves are less likely to move to subsequent phases of clinical tests, and are more likely to succeed in these phases, that is, cash-constrained firms experiment longer. This is also consistent with the fact that long deferments of patent examination are particularly frequent in the pharmaceutical and chemical industries (Henkel and Jell, 2010).[21] In industries characterized by fast learning, cash-constrained firms should invest earlier than unconstrained firms, and have accordingly a lower probability of success. Overall, our model suggests that an empirical analysis on the impact of cash constraints on investment should group firms according both to how financially constrained they are, and to the speed of learning in the industry. Indeed, our results hint that an analysis where firms are grouped

[20] In a hurried investment equilibrium, the good firm is overall less likely to learn bad news, so the total probability of investment is larger (over-investment as compared to the benchmark). In addition, it is less confident about the project upon investing, so investment is riskier, in that the probability of failure is larger (under-experimentation as compared to the benchmark).

[21] Some patent offices allow patent applicants to solicit accelerated and/or deferred examination of their application, lowering or expanding *de facto* the duration of the experimentation period. Patents are often perceived as a way for cash-poor firms to secure financing by signaling their quality (Hall and Harhoff, 2012).

according to their net worth only would possibly underestimate the impact of the cash constraint, by pooling together firms which underinvest and firms which overinvest.

In addition, the model predicts that young firms which are more prone to asymmetric information about the precision of their learning technology behave differently from more established firms for which private information is less of an issue. For instance, young venture capital firms with no established reputation distort the date at which they take ventures public because of signaling concerns. This is reminiscent of Gompers (1996), who evidences the importance of "grandstanding", i.e., the tendency of venture capitalists to take ventures public early, among young venture capital firms. However, while our model clearly predicts such hurried investment, it also suggests that such a phenomenon should be rather prevalent in industries where learning is fast, but less so in industries with slow learning, where we might instead observe the reverse pattern of deferred IPO timing.

Our model also yields predictions in terms of stock price reactions. Since investment possibly reveals information about the entrepreneur's confidence, the stock price should react to investment. However, such a stock price reaction should be different in industries characterized by different learning speeds, and for firms with different financial constraints. In markets where investment is hurried, fast-learning firms invest earlier and investment triggers a positive stock price reaction; conversely, when investment is delayed, the stock price reaction upon investment should be negative, as slow learners invest first. Notice, though, that such a stock price reaction occurs only in the case where the firm holds sufficient cash. Otherwise, the equilibrium is pooling, meaning that the investment date is uninformative, and triggers no price change.

Finally, we evidence two distinct "ranges of inaction" in which investment does not respond to a change in the expected profitability of the project (see Figure 5, where there are two parts on which $\bar{t}$ is flat). One range corresponds to the situation in which the option to wait is not exerted (if investment takes place at date 0, an increment in the NPV

does not change the entrepreneur's behavior, both under complete and incomplete information); the other range corresponds to the zone where there is (full or partial) pooling: the entrepreneur is fully constrained by the incentive problem, and keeps investing at date t_0, even when the NPV marginally increases. Therefore, the non-responsiveness of investment to changes in the value of the investment may have two radically different causes: the option-like feature of investment and the cash constraint. In firms prone to cash constraints and asymmetric information, the non-responsiveness is more likely to originate from an incentive problem, while for firms holding projects which are easier to revert (for instance, when there is a liquid second-hand market for assets), the option has a smaller value, and the non-responsiveness could rather reflect the desire of the entrepreneur to reap the benefits from investment as soon as possible. It would be interesting to test these predictions empirically.

7 Extensions

7.1 More than two types

Given the non-monotonicity of t^* in λ and the result that the direction of the distortion created by signaling concerns depends on the ranking of the efficient investment dates, the generalization to more than two types is not immediate. While a full characterization of equilibria is beyond the scope of this paper, let us try to provide some insights on the direction of the distortion in a fully separating equilibrium in the three-type case.[22] Let $\lambda \in \{\lambda_l, \lambda_m, \lambda_h\}$, with $\lambda_l < \lambda_m < \lambda_h$, and t_l^*, t_m^* and t_h^* denote the efficient investment dates. If $t_l^* < t_m^* < t_h^*$ (resp. $t_h^* < t_m^* < t_l^*$), the logic of the two-type case carries over and all types but the lowest one should delay (resp. hurry) investment. The more interesting situation arises when the complete information investment dates are non-monotone in

[22] We only provide intuitions here. In Section 9.9 of the Appendix, we formally derive necessary conditions which fully characterize the direction of the distortions.

λ. In this case, the analysis yields two distinct insights. First, the incentive constraints of the lowest type drive the direction of the distortions. Actually, the distortion which a given type suffers is the one which would prevail if there were only two types, the lowest type and himself: $t_l^* < t_j^* \Leftrightarrow t_j^* < t_j$ for $j = m, h$. Second, the ranking of the investment dates under asymmetric information has to be the same as under complete information: $t_j^* < t_k^* \Leftrightarrow t_j < t_k$ for all (j, k). Remark that in the two-type case (i.e., when $Pr(\lambda = \lambda_m) = 0$), the fact that $t_l = t_l^*$ together with the first condition implies the second condition.[23] This does not automatically hold with more than two types: actually, making sure that the lowest type does not mimic any of the upper types may either relax the incentive problem among upper types (in which case the second condition is automatically satisfied, as in the two-type case), or worsen it (in which case the second condition kicks in). Specifically, we highlight these different situations by contrasting the following two cases:

1. If $t_h^* < t_l^* < t_m^*$, satisfying the incentive constraints of the low type requires the medium type to delay investment and the high type to hurry investment. By making sure that λ_l does mimic neither λ_m nor λ_h, one therefore relaxes the incentive constraints that λ_m and λ_h do not mimic each other. Accordingly, the complete information ranking between the investment dates of λ_m and λ_h is preserved ($t_h < t_m$). Overall, one has: $t_h < t_h^* < t_l = t_l^* < t_m^* < t_m$.

2. If $t_l^* < t_h^* < t_m^*$, satisfying the incentive constraints of the low type requires both the medium type and the high type to delay investment ($t_h^* < t_h$ and $t_m^* < t_m$.) This may worsen the incentive problem between the medium and high types. In order to ensure incentives between λ_m and λ_h, we must have $t_l^* = t_l < t_h < t_m$.

[23] For instance, when $t_h^* > t_l^*$, the first condition reads $t_h > t_h^*$. Since $t_l = t_l^*$, one has mechanically $t_h > t_l$.

Pushing the logic of the three-type case further, one conjectures that there may not exist separating equilibria with a continuum of types as soon as the efficient investment dates are non-monotonic over the support of types. Indeed, the investment dates should be monotone in a separating equilibrium, which violates the requirement that the ranking of the efficient investment dates be preserved. Remark that, even in the simpler case where the efficient investment dates are monotone in types, establishing the existence of a separating equilibrium formally is challenging, as the usual machinery to show existence of separating equilibrium with a continuum of types cannot be applied because the single-crossing property does not hold (Mailath, 1987; Mailath and von Thadden, 2013).

7.2 Capitalization/depreciation of A

Our results qualitatively go through when A is capitalized at rate $r_0 \leq r$. In this case, we obtain $W(\overline{\lambda}, q, t) - W(\underline{\lambda}, q, t) = Ae^{-(r-r_0)t} f(t)$, a function which is still single-peaked in t, and reaches a maximum at $\tilde{t}_0(r_0)$, where $\tilde{t}_0(r_0)$ increases in r_0. Therefore, the region where delayed investment obtains expands if $r_0 > 0$, and shrinks if $r_0 < 0$.[24] This highlights the complementary role played by the capitalization of the entrepreneur's cash as a sorting force: waiting longer is more of an effective signaling strategy when r_0 is higher, since the amount that is internally financed increases everything else equal, hence the cost incurred by the bad type when mimicking the good type. Notice that this extra effect may generate timing reversals: for instance, when $r_0 > 0$, there are situations where the good type invests earlier than the bad type under complete information, but later under asymmetric information.

[24] See Section 9.10 of the Appendix for a formal analysis.

8 Conclusion

We consider a model in which a cash-constrained entrepreneur learns about the value of a project, but at a speed which is private information. The signaling problem arising from the conjunction of the information friction (private precision of the learning technology) and the financial friction (limited cash) results in the entrepreneur distorting his investment policy when the cash shortage is too severe. This distortion takes the form of hurried investment (under-experimentation) in markets with fast learning, and of delayed investment (over-experimentation) in markets where learning is slower.

A noteworthy property of our model is that the information on the entrepreneur relevant to the market is not his type λ *per se*, but his beliefs about the project, which depend on both his (privately observed) type λ and the (observable) action he takes (the timing decision t). Accordingly, the relevant asymmetric information in the signaling game is the difference between each type's beliefs, which increases and then decreases with time. Such a non-monotonicity explains why signaling may possibly involve hurried or delayed investment. Relatedly, a distinctive feature of our modeling of private information on learning speed is that it is impossible to rank types according to their preference over investment dates: depending on their relative learning speeds, a fast-learning type may be more willing or less willing than a slow-learning type to invest later, that is, the single-crossing property does not hold. While the analysis is made somewhat more complex, it also gets richer. In addition, it highlights in an intuitive way differences in investment timing decisions across firms or industries, by relating these differences to the accuracy of learning in the market.

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9 Appendix

9.1 Proof of Proposition 1

Let us consider the derivative of $-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}$ with respect to λ :

$$\frac{1}{\lambda^2}\ln\frac{p_0 r\,(R-I)}{(1-p_0)\,(\lambda+r)\,I}+\frac{1}{\lambda\,(\lambda+r)}.$$

Its sign is given by the sign of $a(\lambda)=\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{\lambda}{\lambda+r}$.

Note that $a'(\lambda)=-\frac{\lambda}{(\lambda+r)^2}\leq 0$, and that $\lim\limits_{\lambda\to+\infty}a(\lambda)=-\infty$.

We distinguish three cases:

i) $p_0R-I<0$: $-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}>0$ for all λ, and $a(0)=\ln\frac{p_0(R-I)}{(1-p_0)I}<0$.

Therefore, $t^*(\lambda)$ is positive and decreasing for all λ.

ii) $p_0R-I>0$: there exists $\lambda^*=\frac{r(p_0R-I)}{(1-p_0)I}>0$ such that $\lambda\leq\lambda^*\Leftrightarrow t^*(\lambda)=0$. Furthermore, $a(\lambda^*)=\frac{\lambda^*}{\lambda^*+r}>0$, so there also exists $\lambda^{**}>\lambda^*$ such that $a(\lambda^{**})=0$, so $t^*(\lambda)$ is increasing for $\lambda\in[\lambda^*,\lambda^{**}]$, and $t^*(\lambda)$ is decreasing for $\lambda\geq\lambda^{**}$.

iii) $p_0R-I=0$: this implies $t^*(0)=0$ and $t^*(\lambda)>0$ for all $\lambda>0$. One can show that $\lim\limits_{\lambda\to 0}\frac{1}{\lambda^2}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{1}{\lambda(\lambda+r)}=+\infty$.

Therefore, this case is qualitatively similar to the case $p_0R-I>0$: t^* is increasing and then decreasing. □

9.2 D1 beliefs

In this subsection, we examine how D1 restricts beliefs $q(t)$. Let $\overline{W}$ and $\underline{W}$ denote the payoffs of types $\overline{\lambda}$ and $\underline{\lambda}$ in a given equilibrium, and suppose an off-path deviation to t is observed.

Let $\Delta(t)\equiv W(\underline{\lambda},q,t)-\underline{W}-\big(W(\overline{\lambda},q,t)-\overline{W}\big)$ denote the difference between the marginal incentive to deviate to date t for both types, when such a deviation generates

beliefs q.

Using (7), we have $\Delta(t) = \overline{W} - \underline{W} - A\widetilde{f}(t)$, which is independent of q.

D1 imposes to prune $\overline{\lambda}$, i.e. consider $q(t) = 0$, if $\Delta(t) > 0$, and to prune $\underline{\lambda}$, i.e. consider $q(t) = 1$, if $\Delta(t) < 0$. If $\Delta(t) = 0$, we assume that $q(t) = 1$ for simplicity, but this is innocuous. Overall, D1 imposes that, for any t off the equilibrium path, the following holds:

$$q(t) \in \{0, 1\} \text{ and } q(t) = 1 \Leftrightarrow \Delta(t) = \overline{W} - \underline{W} - A\widetilde{f}(t) \leq 0.$$

9.3 Proof of Lemma 1

Consider the case $\overline{t}^* \geq t_0$.

Suppose that there is a separating equilibrium $(\underline{t}^*, \overline{t})$ such that $\overline{t}^* < \overline{t}$. Since $\overline{t}$ must satisfy (9b), we have

$$W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \overline{t}) = W(\overline{\lambda}, 1, \overline{t}) - A\widetilde{f}(\overline{t}).$$

Let us denote $\alpha(\overline{t}) \equiv W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, \overline{t}) + A\widetilde{f}(\overline{t}) \geq 0$ the "slack" in the incentive constraint of the bad type.

One can write

$$\Delta(t) = W(\overline{\lambda}, 1, \overline{t}) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t) = A[\widetilde{f}(\overline{t}) - \widetilde{f}(t)] - \alpha(\overline{t}).$$

Since $\overline{t} > \overline{t}^* \geq t_0$, there exists $\epsilon > 0$ such that $\overline{t} - \epsilon \geq \overline{t}^* \geq t_0$. One has $\Delta(\overline{t} - \epsilon) < 0$, which implies $q(\overline{t} - \epsilon) = 1$. In addition, we must have $W(\overline{\lambda}, 1, \overline{t} - \epsilon) > W(\overline{\lambda}, 1, \overline{t})$ because $\overline{t}^* \leq \overline{t} - \epsilon \leq \overline{t}$. Therefore, $\overline{t} - \epsilon$ is a profitable deviation for $\overline{\lambda}$, so $(\underline{t}^*, \overline{t})$ cannot be an equilibrium.

Suppose now that $\overline{t} < t_0 \leq \overline{t}^*$. By the same mechanics, one shows that type $\overline{\lambda}$ can strictly increase his payoff by deviating to $\overline{t} + \epsilon$ such that $\overline{t} + \epsilon \leq t_0 \leq \overline{t}^*$.

The proof is similar in the case $t_0 > \overline{t}^*$. $\square$

Before we go further, let us notice that

$$W^*(\overline{\lambda}, t) = W^*(\underline{\lambda}, t) + I\widetilde{f}(t). \tag{10}$$

We derive from this that

$$\overline{t}^* \leq t_0 \Leftrightarrow \underline{t}^* \leq \overline{t}^*. \tag{11}$$

Put differently, $\overline{t}^*$ is always between t_0 and $\underline{t}^*$.

9.4 Proof of Proposition 2

Let us first look for conditions for investment at the optimal dates $(\underline{t}^*, \overline{t}^*)$ to be an equilibrium. It is clear that (9a) and (9c) are satisfied for $\overline{t} = \overline{t}^*$, as type $\overline{\lambda}$ gets his complete information payoff. One can also show that (9d) holds for $\overline{t} = \overline{t}^*$. Suppose it does not, i.e., there is some $t_a \notin \left\{\overline{t}^*, \underline{t}^*\right\}$ such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$.

Since D1 imposes to have $q(t) \in \{0, 1\}$, we must have $q(t_a) = 1$, which implies:

$$\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^*) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t_a) \leq 0.$$

Therefore, we have

$$W(\underline{\lambda}, 1, t_a) = W(\overline{\lambda}, 1, t_a) - A\widetilde{f}(t_a) < W(\overline{\lambda}, 1, \overline{t}^*) - A\widetilde{f}(t_a) \leq W(\underline{\lambda}, 0, \underline{t}^*).$$

A contradiction.

Consequently, (9b) is a necessary and sufficient condition for $(\underline{t}^*, \overline{t}^*)$ to be an equilibrium. That (9b) holds at $\overline{t} = \overline{t}^*$ reads

$$W(\underline{\lambda}, 1, \overline{t}^*) = W(\overline{\lambda}, 1, \overline{t}^*) - A\widetilde{f}(\overline{t}^*) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{12}$$

If (12) holds for some $\tilde{A}$, then it must hold for all $A > \tilde{A}$. Note also that (12) does not hold for $A = 0$. When $A \to I$, $W(\underline{\lambda}, 1, \overline{t}^*) \to W(\underline{\lambda}, 0, \overline{t}^*) < W(\underline{\lambda}, 0, \underline{t}^*)$, so (12) holds. We conclude that there exists $A_0 \in (0, I)$ such that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium iff $A \geq A_0$, with

$$A_0 = \frac{W^*\left(\overline{\lambda}, \overline{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{\widetilde{f}(\overline{t}^*)}$$

Let us now suppose that

$$A < A_0 \Leftrightarrow W(\underline{\lambda}, 1, \overline{t}^*) > W(\underline{\lambda}, 0, \underline{t}^*).$$

Before going further, let us remark that the function $t \mapsto W(\overline{\lambda}, 1, t) - A\widetilde{f}(t)$ is increasing on $[t_0, \overline{t}^*]$ if $t_0 < \overline{t}^*$, and decreasing on $[\overline{t}^*, t_0]$ if $\overline{t}^* < t_0$. This implies that

$$\forall t \in [\min\left(t_0, \overline{t}^*\right), \max\left(t_0, \overline{t}^*\right)],\ W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\overline{\lambda}, 1, t) - A\widetilde{f}(t). \tag{13}$$

We now establish the following lemma:

Lemma 2 *A separating equilibrium exists if and only if* $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$.

Proof Let us first prove that $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$ is a necessary condition for a separating equilibrium.

Suppose we have $W(\underline{\lambda}, 1, t_0) > W(\underline{\lambda}, 0, \underline{t}^*)$, with $W(\underline{\lambda}, 1, t_0) = W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0)$.

Using (13), we derive that

$$\forall t \in [\min\left(t_0, \overline{t}^*\right), \max\left(t_0, \overline{t}^*\right)],\ W(\overline{\lambda}, 1, t) - A\widetilde{f}(t) > W(\underline{\lambda}, 0, \underline{t}^*).$$

From Lemma 1, we derive that there is no separating equilibrium.

Let us now show that $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$ is a sufficient condition for a separating equilibrium. Suppose $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*) < W(\underline{\lambda}, 1, \overline{t}^*)$. This implies:

- If $t_0 < \overline{t}^* : \exists! \overline{t}^h \in [t_0, \overline{t}^*], W(\underline{\lambda}, 1, \overline{t}^h) = W(\overline{\lambda}, 1, \overline{t}^h) - A\widetilde{f}(\overline{t}^h) = W(\underline{\lambda}, 0, \underline{t}^*)$,

- If $\overline{t}^* < t_0 : \exists! \overline{t}^d \in [\overline{t}^*, t_0], W(\underline{\lambda}, 1, \overline{t}^d) = W(\overline{\lambda}, 1, \overline{t}^d) - A\widetilde{f}(\overline{t}^d) = W(\underline{\lambda}, 0, \underline{t}^*)$,

Let us now show that the good type has no profitable deviation starting from a candidate equilibrium $\overline{t} = \overline{t}^h$ (resp. $\overline{t}^d$), i.e., (9a) and (9c) are satisfied.

Before doing so, notice, using (11), that $t_0 < \overline{t}^* \Rightarrow t_0 < \overline{t}^h < \overline{t}^* < \underline{t}^*$ and $t_0 > \overline{t}^* \Rightarrow t_0 > \overline{t}^d > \overline{t}^* > \underline{t}^*$. For $i \in \{h, d\}$, one has therefore $\tilde{f}(\overline{t}^i) > \tilde{f}(\underline{t}^*)$.

Let us first check that the good type has no incentives to mimic the bad type, i.e., invest at $\underline{t}^*$.

For $i \in \{h, d\}$, one has $W(\underline{\lambda}, 1, \overline{t}^i) \geq W(\underline{\lambda}, 0, \underline{t}^*) \Leftrightarrow W(\overline{\lambda}, 1, \overline{t}^i) \geq W(\underline{\lambda}, 0, \underline{t}^*) + A\tilde{f}(\overline{t}^i)$. Since $\tilde{f}(\overline{t}^i) > \tilde{f}(\underline{t}^*)$, we derive $W(\overline{\lambda}, 1, \overline{t}^i) \geq W(\underline{\lambda}, 0, \underline{t}^*) + A\tilde{f}(\underline{t}^*) = W(\overline{\lambda}, 0, \underline{t}^*)$.

Let us now check off-path deviations. Consider first the case $t_0 < \overline{t}^*$. Suppose the good type deviates from $\overline{t} = \overline{t}^h$ to some date t_a :

- If $t_a < \overline{t}^h$, we have $W(\overline{\lambda}, q(t_a), t_a) \leq W(\overline{\lambda}, 1, t_a) < W(\overline{\lambda}, 1, \overline{t}^h)$. So such a deviation cannot be profitable for any out-of-equilibrium beliefs $q(t_a)$.

- If $\overline{t}^h < t_a$: one can write $\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^h) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t_a) = A[\widetilde{f}(\overline{t}^h) - \widetilde{f}(t_a)] > 0$ for $t_0 \leq \overline{t}^h < t_a$. Therefore, we must have $q(t_a) = 0$. The benefit from deviating to t_a becomes $W(\overline{\lambda}, 0, t_a) - W(\overline{\lambda}, 1, \overline{t}^h) = W(\underline{\lambda}, 0, t_a) + A\widetilde{f}(t_a) - [W(\underline{\lambda}, 1, \overline{t}^h) + A\widetilde{f}(\overline{t}^h)] = A[\widetilde{f}(t_a) - \widetilde{f}(\overline{t}^h)] + W(\underline{\lambda}, 0, t_a) - W(\underline{\lambda}, 0, \underline{t}^*) < 0$, so this deviation is not profitable.

In the case $\overline{t}^* < t_0$, the proof is similar.

Therefore, the good type has no profitable deviation.

The last thing we need to show is that the bad type cannot benefit from a deviation off path either. Suppose there exists t_a such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since D1

imposes to have $q(t) \in \{0,1\}$ for all t in a separating equilibrium, we must have $q(t_a) = 1$, which implies

$$\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^i) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t_a) \leq 0 \Leftrightarrow \widetilde{f}(\overline{t}^i) \leq \widetilde{f}(t_a).$$

For $i = h$, $\widetilde{f}(\overline{t}^h) \leq \widetilde{f}(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^h)$. Indeed, given $\overline{t}^h \geq t_0$, a necessary condition for $\widetilde{f}(\overline{t}^h) \leq \widetilde{f}(t_a)$ is $t_a \leq \overline{t}^h$.

Similarly, $\widetilde{f}(\overline{t}^d) \leq \widetilde{f}(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^d)$ in the case $\overline{t}^* < t_0$.

Therefore, one has

$$\begin{aligned} W(\underline{\lambda}, 1, t_a) &= W(\overline{\lambda}, 1, t_a) - A\widetilde{f}(t_a) \\ &\leq W(\overline{\lambda}, 1, \overline{t}^i) - A\widetilde{f}(\overline{t}^i) \\ &= W(\underline{\lambda}, 1, \overline{t}^i) \\ &= W(\underline{\lambda}, 0, \underline{t}^*). \end{aligned}$$

This contradicts $W(\underline{\lambda}, 1, t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$, so the bad type has no profitable deviation. □

Let us finally check under which conditions we have $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$.

One can rewrite the condition as

$$W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{14}$$

Again, if $\tilde{A}$ satisfies (14), then so does $A > \tilde{A}$. Furthermore, (14) holds for $A = A_0$.

Indeed, (14) can be rewritten

$$W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\overline{\lambda}, 1, \overline{t}^*) - A_0\widetilde{f}(\overline{t}^*).$$

From $W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\overline{\lambda}, 1, \overline{t}^*) - A\widetilde{f}(\overline{t}^*)$, it is then clear that (14) holds for $A = A_0$.

Therefore, there exists $A_1 \leq A_0$ such that a separating equilibrium exists if and only if $A \geq A_1$. We have:

$$A_1 = \max\left(0, \frac{W^*(\overline{\lambda}, t_0) - W^*(\underline{\lambda}, \underline{t}^*)}{\widetilde{f}(t_0)}\right)$$

Finally, notice that there may be other separating equilibria involving $\overline{t} \in [t_0, \overline{t}^h)$ (resp. $\overline{t} \in (\overline{t}^d, t_0]$), but they would give a strictly lower profit to the good type than t^h (resp. t^d). □

9.5 Proof of Proposition 3

Let us first establish the following lemma:

Lemma 3 *In any non-separating equilibrium, $\overline{\lambda}$ invests at date $\overline{t} = t_0$ with probability* 1. *In addition, $\underline{\lambda}$ either invests at t_0 with probability* 1 *(pooling equilibrium), or randomizes between investing at t_0 and $\underline{t}^*$ (semi-pooling equilibrium).*

Proof Let $\mathbb{T}$ be the set of dates at which both types invest with positive probability. Given that D1 imposes to consider $q(t) \in \{0, 1\}$ for each t off path, we have that $q(t) \in (0, 1) \Leftrightarrow t \in \mathbb{T}$.

We first establish that $\mathbb{T}$ has at most two elements. Indeed, suppose $\mathbb{T}$ has at least three distinct elements (t_a, t_b, t_c). By definition of $\mathbb{T}$, one has

$$W(\lambda, q(t_a), t_a) = W(\lambda, q(t_b), t_b) = W(\lambda, q(t_c), t_c) \text{ for } \lambda \in \{\underline{\lambda}, \overline{\lambda}\}. \tag{15}$$

Using (7), one derives that $\widetilde{f}(t_a) = \widetilde{f}(t_b) = \widetilde{f}(t_c)$, which is impossible, since $\widetilde{f}$ is continuous and single-peaked.

Suppose now that $\mathbb{T}$ has two distinct elements (t_a, t_b). One at least is different from t_0, say t_a. We then have $\Delta(t) = W(\overline{\lambda}, q(t_a), t_a) - W(\underline{\lambda}, q(t_a), t_a) - A\widetilde{f}(t) = A[\widetilde{f}(t_a) - \widetilde{f}(t)]$, using (7). If $t_a < t_0$, we have $\Delta(t_a + \epsilon) < 0$, so $q(t_a + \epsilon) = 1$. Since $q(t_a) < 1$, one can

always find ϵ small enough such that $W(\lambda, 1, t_a + \epsilon) > W(\lambda, q(t_a), t_a)$ for all λ. So there is a profitable deviation. The same reasoning holds for $t_a > t_0$.

We conclude that $\mathbb{T}$ is a singleton. If the unique element of $\mathbb{T}$ is not t_0, there is always a profitable deviation, by the same reasoning as above. Therefore, $\mathbb{T} = \{t_0\}$.

Suppose now that $\overline{\lambda}$ invests with positive probability at some $t_a \neq t_0$. Since $\mathbb{T} = \{t_0\}$, $\underline{\lambda}$ must then invest with probability zero at date t_a.

We therefore have

$$W(\overline{\lambda}, 1, t_a) = W(\overline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, q(t_0), t_0) + A\widetilde{f}(t_0).$$

This implies that

$$W(\underline{\lambda}, 1, t_a) = W(\overline{\lambda}, 1, t_a) - A\widetilde{f}(t_a) = W(\underline{\lambda}, q(t_0), t_0) + A[\widetilde{f}(t_0) - \widetilde{f}(t_a)] > W(\underline{\lambda}, q(t_0), t_0).$$

So type $\underline{\lambda}$ then strictly prefers to invest at date t_a than at t_0. A contradiction. □

Let us now turn to the proof of the Proposition. Suppose first that a separating equilibrium exists, i.e., $A \geq A_1$. Using Lemma 2, this is equivalent to $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$. We then have:

$$W(\underline{\lambda}, q(t_0), t_0) < W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*).$$

Therefore, the bad type cannot invest at t_0 with positive probability, as this is strictly dominated by investing at $\underline{t}^*$. Therefore, there are neither semi-pooling nor pooling equilibria.

Before we derive equilibrium conditions for a pooling and a semi-pooling equilibrium, notice that, since the equilibrium payoffs are $W(\lambda, q(t_0), t_0)$ for each type λ, we have $\Delta(t) = A[\widetilde{f}(t_0) - \widetilde{f}(t)] > 0$, so any off-path deviation generates beliefs $q(t) = 0$.

Conditions for a pooling equilibrium The following conditions must be satisfied for a pooling equilibrium $\underline{t} = \overline{t} = t_0$ to exist:

$$W(\underline{\lambda}, q_0, t_0) \geq W(\underline{\lambda}, 0, \underline{t}^*) \tag{16}$$

$$W(\overline{\lambda}, q_0, t_0) \geq W(\overline{\lambda}, 0, t) \text{ for all } t \neq t_0. \tag{17}$$

A necessary condition for a pooling equilibrium is that a separating equilibrium does not exist, i.e., $W(\underline{\lambda}, 1, t_0) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since $W(\underline{\lambda}, q, t_0)$ is increasing in q, and since $W(\underline{\lambda}, 0, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$, there exists a critical value of $\overline{q}$ such that (16) holds if and only if $q_0 \geq \overline{q}$. $\overline{q}$ satisfies

$$W(\underline{\lambda}, \overline{q}, t_0) = W(\underline{\lambda}, 0, \underline{t}^*).$$

Let $\hat{t} \in \underset{t}{\arg\max}\ W(\overline{\lambda}, 0, t)$:

$$\begin{aligned} W(\overline{\lambda}, \overline{q}, t_0) - W(\overline{\lambda}, 0, \hat{t}) &= W(\underline{\lambda}, \overline{q}, t_0) + A\widetilde{f}(t_0) - W(\underline{\lambda}, 0, \hat{t}) - A\widetilde{f}(\hat{t}) \\ &= W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, \hat{t}) + A\left[\widetilde{f}(t_0) - \widetilde{f}(\hat{t})\right] \\ &> 0. \end{aligned}$$

This implies that (16) $\Rightarrow$ (17). Consequently, there is a pooling equilibrium in which all types invest at $t = t_0$ if and only if $q_0 \geq \overline{q}$. To derive this equilibrium condition as a function of A, let us first notice that, at $A = A_1$, we have $\overline{q} = 1 > q_0$, by definition of A_1. So, the pooling equilibrium does not exist when A is sufficiently close to A_1. Furthermore, one sees that (6) implies that

$$W(\underline{\lambda}, q, t) = W(\underline{\lambda}, 0, t) + q\widetilde{f}(t)(I - A),$$

which gives

$$\overline{q} = \frac{W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, t_0)}{\widetilde{f}(t_0)(I - A)}.$$

Therefore, $\overline{q}$ is increasing in A. When $A = 0$, we have $\overline{q} = \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{\widetilde{f}(t_0)I}$. We derive that:

- if $q_0 \leq \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{I\widetilde{f}(t_0)}$, no pooling equilibrium ever exists,

- if $q_0 > \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{I\widetilde{f}(t_0)}$, a pooling equilibrium exists if and only if A is small enough.

Overall, a pooling equilibrium exists if and only if $A \leq A_2$, with

$$A_2 = \max\left(0, I - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0\widetilde{f}(t_0)}\right).$$

Conditions for a semi-pooling equilibrium The following conditions must be satisfied for a semi-pooling equilibrium to exist:

$$W(\underline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, 0, \underline{t}^*) \tag{18}$$

$$W(\overline{\lambda}, q(t_0), t_0) > W(\overline{\lambda}, 0, t) \text{ for all } t. \tag{19}$$

Using the same argument as above, it is easy to see that (18) $\Rightarrow$ (19).

Finally, observe that $q(t_0) > q_0$, because $\underline{\lambda}$ does not invest at t_0 with probability 1, whereas $\overline{\lambda}$ does. So, if $q_0 \geq \overline{q} \Leftrightarrow A \leq A_2$, (18) cannot hold. Conversely, if $q_0 < \overline{q}$, type $\underline{\lambda}$ can always invest at date t_0 with a probability $x \in (0, 1)$ such that $q(t_0) = \frac{q_0}{q_0+(1-q_0)x} = \overline{q}$, in which case (18) is satisfied. Hence the result. □

9.6 Proof of Proposition 4

- If $A \geq A_0$, $\overline{t} = \overline{t}^*$, which is independent of A,

- If $A < A_1$, $\overline{t} = t_0$, which is independent of A,

- If $A \in [A_1, A_0)$, in the least cost separating equilibrium, the good type invests at

date $\overline{t}^i$ given by:

$$W(\underline{\lambda}, 1, \overline{t}^i) = W(\overline{\lambda}, 1, \overline{t}^i) - A\widetilde{f}(\overline{t}^i) = W(\underline{\lambda}, 0, \underline{t}^*). \tag{20}$$

Differentiating with respect to A yields

$$\frac{\partial \overline{t}^i}{\partial A} = \frac{\widetilde{f}(\overline{t}^i)}{W_3\left(\overline{\lambda}, 1, \overline{t}^i\right) - A\frac{\partial \widetilde{f}}{\partial t}(\overline{t}^i)}.$$

The numerator is positive. The denominator is negative for $i = d$, as $\overline{t}^d \in \left[\overline{t}^*, t_0\right]$, and positive for $i = h$, since $\overline{t}^h \in \left[t_0, \overline{t}^*\right]$. This implies $\frac{\partial \overline{t}^h}{\partial A} > 0$, and $\frac{\partial \overline{t}^d}{\partial A} < 0$, so $\frac{\partial |\overline{t}^* - \overline{t}|}{\partial A} < 0$. □

9.7 Proof of Proposition 5

Let us now show that reversals can arise in the case where there is a separating equilibrium, i.e., when $A \geq A_1$. If $A \leq A_2$, both types invest at t_0 so the good type is more optimistic upon investing. If $A \in (A_2, A_1)$, the bad type randomizes, so that the occurrence of a belief reversal depends on the realized investment date. For simplicity, let us then only focus on situations where both types play different pure strategies. In this case, $\underline{t} = \underline{t}^*$. We look for the sign of $p^*(\overline{\lambda}, \overline{t}) - p^*(\underline{\lambda}, \underline{t}^*)$, which has the same sign as $e^{-\underline{\lambda}\underline{t}^*} - e^{\overline{\lambda}\overline{t}}$.

Notice first that, under complete information, this difference reads

$$e^{-\underline{\lambda}\underline{t}^*} - e^{-\overline{\lambda}\overline{t}^*} = \frac{rp_0(R-I)}{(1-p_0)I}\left(\frac{1}{\underline{\lambda}+r} - \frac{1}{\overline{\lambda}+r}\right) > 0.$$

Since the good type is more optimistic than the bad type upon investing in the complete information case, he is a fortiori more confident when investment is delayed. Therefore, let us focus on the case where investment is hurried. In this case, $\overline{t}$ is increasing in A (see Proposition 4), which implies that $e^{-\underline{\lambda}\underline{t}^*} - e^{-\overline{\lambda}\overline{t}}$ is increasing in A as well. For $A \geq A_0$, investment is efficient, and there is no reversal.

There are two situations then:

- If there is no belief reversal at $A = A_1$, then there is never belief reversal.
- If there is belief reversal at $A = A_1$, then there exists $\underline{A} \in (A_1, A_0)$ such that a belief reversal occurs if $A_1 \leq A < \underline{A}$.

We complete the proof by showing the following lemmas.

Lemma 4 *Suppose that $\overline{t}^* < \underline{t}^*$ (investment is hurried).*

Then $\exists! \lambda_a \in (0, \overline{\lambda})$ such that $A_1 = 0 \Leftrightarrow \underline{\lambda} \geq \lambda_a$.

Proof First, $W(\underline{\lambda}, 0, \underline{t}^*)$ is increasing in $\underline{\lambda}$, using the envelope theorem. Second, $W(\overline{\lambda}, 1, t_0)$ is decreasing in $\underline{\lambda}$, as $W(\overline{\lambda}, 1, t)$ is increasing in t at $t = t_0$ ($\overline{t}^* < \underline{t}^* \Rightarrow t_0 < \overline{t}^*$), and t_0 is decreasing in $\underline{\lambda}$. Therefore, $W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, t_0)$ is increasing in $\underline{\lambda}$.

When $\underline{\lambda} \to \overline{\lambda}$, we have $\lim_{\underline{\lambda} \to \overline{\lambda}} W(\overline{\lambda}, 1, t_0) = W(\overline{\lambda}, 1, \frac{1}{\overline{\lambda}+r}) < \lim_{\underline{\lambda} \to \overline{\lambda}} W(\underline{\lambda}, 0, \underline{t}^*) = W(\overline{\lambda}, 1, \overline{t}^*)$.

- When $p_0 R - I \geq 0$, $\overline{t}^* < \underline{t}^* \Leftrightarrow \overline{\lambda} > \lambda^{**}$ and $\underline{\lambda} > \underline{\lambda}_0$.

  At $\underline{\lambda} = \underline{\lambda}_0$, we have by definition $\underline{t}^* = t_0$, so $W(\underline{\lambda}, 0, \underline{t}^*) < W(\overline{\lambda}, 1, t_0)$.

- When $p_0 R - I < 0$, one always has $\overline{t}^* < \underline{t}^*$.

  At $\underline{\lambda} = 0$, we have $W(\underline{\lambda}, 0, \underline{t}^*) = 0 < W(\overline{\lambda}, 1, t_0)$

In either case, there exists a $\lambda_a < \overline{\lambda}$ such that

$$A_1 = 0 \Leftrightarrow W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, t_0) \geq 0 \Leftrightarrow \underline{\lambda} \geq \lambda_a.$$

□

Lemma 5 *Suppose that $\overline{t}^* < \underline{t}^*$ and $A = A_1$.*

Then there exists λ_1 such that

$$p(\overline{\lambda}, \overline{t}) - p(\underline{\lambda}, \underline{t}^*) < 0 \Leftrightarrow \underline{\lambda} > \lambda_1.$$

Proof We distinguish two cases:

1. $\underline{\lambda} \geq \lambda_a \Leftrightarrow A_1 = 0$. In this case, we have

$$W(\overline{\lambda}, 1, \overline{t}) = W(\underline{\lambda}, 0, \underline{t}^*)$$

$$\Leftrightarrow e^{-r\overline{t}} \left(R - \frac{I}{p(\overline{\lambda}, \overline{t})} \right) = e^{-r\underline{t}^*} \left(R - \frac{I}{p(\underline{\lambda}, \underline{t}^*)} \right).$$

   This implies

$$\frac{I}{p(\underline{\lambda}, \underline{t}^*)} - \frac{I}{p(\overline{\lambda}, \overline{t})} = \left(R - \frac{I}{p(\overline{\lambda}, \overline{t})} \right) \left(1 - e^{r(\underline{t}^* - \overline{t})} \right) < 0,$$

   where the last inequality derives from $p(\overline{\lambda}, \overline{t})R - I > 0$ and $\overline{t} < \overline{t}^* < \underline{t}^*$.

   Therefore, if the equilibrium is separating at $A = 0$, it must involve a reversal.

2. $\underline{\lambda} < \lambda_a \Leftrightarrow A_1 > 0$. In this case, one has $\overline{t} = t_0$ at $A = A_1$ by definition.

   Therefore, we are interested in the sign of $p^*(\overline{\lambda}, t_0) - p^*(\underline{\lambda}, \underline{t}^*)$. This function decreases in $\underline{\lambda}$, as t_0 decreases in $\underline{\lambda}$.

   By continuity, one has $p^*(\overline{\lambda}, t_0) - p^*(\underline{\lambda}, \underline{t}^*) < 0$ at $\underline{\lambda} = \lambda_a$.

   - When $p_0 R - I \geq 0$, $\overline{\lambda} > \lambda^{**}$, and $\underline{\lambda} > \underline{\lambda}_0$, one has $t_0 = \underline{t}^*$ at $\underline{\lambda} = \underline{\lambda}_0$.

     This implies $p^*(\overline{\lambda}, t_0) - p^*(\underline{\lambda}, \underline{t}^*) > 0$.

   - When $p_0 R - I < 0$, one has $p^*(\overline{\lambda}, t_0) - p^*(\underline{\lambda}, \underline{t}^*) = p^*(\overline{\lambda}, t_0) - p_0 > 0$ at $\underline{\lambda} = 0$.

   In either case, there exists a unique $\lambda_1 < \lambda_a$ such that $p^*(\overline{\lambda}, t_0) - p^*(\underline{\lambda}, \underline{t}^*) \leq 0 \Leftrightarrow \underline{\lambda} \geq \lambda_1$. □

9.8 Proof of Proposition 6

Before deriving how the equilibrium varies with p_0, let us examine how $\overline{t}^*, \underline{t}^*$ and t_0 compare when p_0 varies. Recall first that

$$t^*(\lambda) = \max\left(-\frac{1}{\overline{\lambda}}\ln\frac{p_0 r\,(R-I)}{(1-p_0)\,(\lambda+r)\,I}, 0\right).$$

Therefore, $t^*(\lambda) > 0 \Leftrightarrow p_0 < \frac{(\lambda+r)I}{rR+\lambda I}$.

Let us consider how $\underline{t}^* - \overline{t}^*$ varies with p_0.

For $p_0 \geq \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, $\underline{t}^* = 0$, so $\underline{t}^* - \overline{t}^* \leq 0$. For $p_0 < \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, we have

$$\underline{t}^* - \overline{t}^* = (\frac{1}{\overline{\lambda}} - \frac{1}{\underline{\lambda}})\ln\frac{p_0}{1-p_0} + \frac{1}{\overline{\lambda}}\ln\frac{r(R-I)}{(\overline{\lambda}+r)I} - \frac{1}{\underline{\lambda}}\ln\frac{r(R-I)}{(\underline{\lambda}+r)I}.$$

This function is decreasing in p_0, tends to $+\infty$ when $p_0 \to 0$, and is negative at $p_0 = \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, so there exists a $\hat{p} < \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$ such that $\overline{t}^* \leq \underline{t}^* \Leftrightarrow p_0 \leq \hat{p}$. In addition, since t_0 does not depend on p_0, $\overline{t}^* \geq t_0 \Leftrightarrow p_0 \leq \hat{p}$.

Let us define the following function:

$$g(p_0) = W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 1, \overline{t}^*).$$

We know that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium if and only if $g(p_0) \geq 0$. Instead of working with p_0, it will prove easier to work with

$$x(p_0) \equiv \frac{p_0 r(R-I)}{(1-p_0)(\overline{\lambda}+r)I}. \tag{21}$$

Let $\tilde{g}$ be such that $g = \tilde{g} \circ x$. There are three different cases to be considered:

1. If $p_0 \leq \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I} \Leftrightarrow x < \frac{\underline{\lambda}+r}{\overline{\lambda}+r}$, then $\underline{t}^* = -\frac{1}{\underline{\lambda}}\ln(\frac{\overline{\lambda}+r}{\underline{\lambda}+r}x) > 0$, and $\overline{t}^* = -\frac{1}{\overline{\lambda}}\ln x > 0$. In this

case, $\tilde{g}(x) = \frac{R-I}{r(R-I)+(\overline{\lambda}+r)xI}\phi(x)$, with

$$\phi(x) = \left(\frac{\overline{\lambda}+r}{\underline{\lambda}+r}\right)^{\frac{r}{\underline{\lambda}}+1} \underline{\lambda} I x^{\frac{r}{\underline{\lambda}}+1} - \left(\overline{\lambda} I + rA\right) x^{\frac{r}{\overline{\lambda}}+1} + Arx^{\frac{r+\underline{\lambda}}{\overline{\lambda}}}. \tag{22}$$

2. If $\frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I} < p_0 < \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I} \Leftrightarrow \frac{\underline{\lambda}+r}{\overline{\lambda}+r} < x < 1$, then $\underline{t}^* = 0$ and $\overline{t}^* = -\frac{1}{\overline{\lambda}}\ln x > 0$. In this case, we have $\tilde{g}(x) = \frac{R-I}{r(R-I)+(\overline{\lambda}+r)xI}\psi(x)$, with

$$\psi(x) = \left(\overline{\lambda}+r\right) xI - rI - \left(\overline{\lambda} I + rA\right) x^{\frac{r}{\overline{\lambda}}+1} + Arx^{\frac{r+\underline{\lambda}}{\overline{\lambda}}}. \tag{23}$$

3. If $p_0 \geq \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I} \Leftrightarrow x \geq 1$, then $\overline{t}^* = \underline{t}^* = 0$. In this case, we have $\tilde{g}(x) = 0$.

Let $\hat{x} \equiv x(\hat{p}) = (\frac{\underline{\lambda}+r}{\overline{\lambda}+r})^{\frac{\overline{\lambda}}{\overline{\lambda}-\underline{\lambda}}} < \frac{\underline{\lambda}+r}{\overline{\lambda}+r}$. It is easy to see that $g(\hat{p}) < 0$, which is equivalent to $\tilde{g}(\hat{x}) < 0$.

Let us start with the first case, and consider the function ϕ defined in (22). On can rewrite $\phi(x) = \alpha x^a + \beta x^b + \gamma x^c$, with $a = \frac{r}{\underline{\lambda}} + 1 > b = \frac{r}{\overline{\lambda}} + 1 > c = \frac{r+\underline{\lambda}}{\overline{\lambda}}$, and $\alpha > 0, \beta < 0, \gamma > 0$. Observe that $x \mapsto x^{-c}\phi(x)$ is decreasing and then increasing as x varies from 0 to $+\infty$. This function is strictly positive when $x = 0$ and tends to $+\infty$ when x tends to $+\infty$.

In the second case, let us consider the function ψ defined in (23). Simple algebra allows us to show that ψ is increasing and then decreasing on $\mathbb{R}$. When $x \to 1$, $\tilde{g}$ tends to zero by continuity and $\psi'(1) < 0$.

Since we know that $\phi(\hat{x}) < 0$, there are only two possible cases:

- $\phi(\frac{\underline{\lambda}+r}{\overline{\lambda}+r}) = \psi(\frac{\underline{\lambda}+r}{\overline{\lambda}+r}) < 0$: in this case, there exists a unique $x_0 \in [0, \frac{\underline{\lambda}+r}{\overline{\lambda}+r}]$ such that $\phi(x_0) = 0$ and a unique $x_1 \in [\frac{\underline{\lambda}+r}{\overline{\lambda}+r}, 1]$ such that $\psi(x_1) = 0$.

- $\phi(\frac{\underline{\lambda}+r}{\overline{\lambda}+r}) = \psi(\frac{\underline{\lambda}+r}{\overline{\lambda}+r}) > 0$: there exists a unique (x_0, x_1) with $0 < x_0 < x_1 < \frac{\underline{\lambda}+r}{\overline{\lambda}+r}$ such that $\phi(x_0) = \phi(x_1) = 0$. In addition, we have $\psi(x) > 0$ for all $x \in (\frac{\underline{\lambda}+r}{\overline{\lambda}+r}, 1)$.

In any case, we have proven that $\tilde{g}(x) \geq 0$ if and only if $x \leq x_0$ or $x \geq x_1$. Consequently, there exist $(\underline{\underline{p}}, \overline{\overline{p}}) \in [0, \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I}]^2$ with $\underline{\underline{p}} < \hat{p} < \overline{\overline{p}}$ such that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium if and only if $p_0 \notin (\underline{\underline{p}}, \overline{\overline{p}})$.

Let us now find conditions under which a separating equilibrium exists. From Lemma 2, we know that a separating equilibrium exists if and only if $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*) \Leftrightarrow h(p_0) \geq 0$, where $h(p_0) \equiv W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 1, t_0)$. It is straightforward to show that h is convex in p_0. Because $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 1, \overline{t}^*)$, we have $h(p_0) \geq g(p_0)$ for all p_0 (with equality only at $p_0 = \hat{p}$). Therefore, $h(\underline{\underline{p}}) > g(\underline{\underline{p}}) = 0$ and $h(\overline{\overline{p}}) > g(\overline{\overline{p}}) = 0$ Finally, $h(\hat{p}) = g(\hat{p}) < 0$, so there exists a unique $(\underline{p}, \overline{p})$ such that $h(\underline{p}) = h(\overline{p}) = 0$. In addition, one has $\underline{\underline{p}} < \underline{p} < \hat{p} < \underline{p} < \overline{\overline{p}}$.

Therefore the equilibrium investment date $\overline{t}$ is

- $\overline{t}^*$ if $p_0 \leq \underline{\underline{p}}$ (efficient investment date),
- $\overline{t}^h < \overline{t}^*$ if $\underline{\underline{p}} < p_0 < \underline{p}$ (separating equilibrium, hurried investment),
- $t_0 \leq \overline{t}^*$ if $\underline{p} \leq p_0 \leq \hat{p}$ (pooling equilibrium, hurried investment),
- $t_0 > \overline{t}^*$ if $\hat{p} < p_0 < \overline{p}$ (pooling equilibrium, delayed investment),
- $\overline{t}^d > \overline{t}^*$ if $\overline{p} < p < \overline{\overline{p}}$ (separating equilibrium, delayed investment),
- $\overline{t}^*$ if $p_0 \geq \overline{\overline{p}}$ (efficient investment date). □

9.9 The three-type case

Preliminaries

Let $W(\lambda_i, \lambda_j, t)$ denote the expected payoff at date 0 of type λ_i from investing at date t and being perceived by the market as type λ_j.

Remark that

$$W(\lambda_i, \lambda_j, t) - W(\lambda_k, \lambda_j, t) = A\tilde{f}(\lambda_i, \lambda_k, t) \tag{24}$$

where

$$\tilde{f}(\lambda_i, \lambda_k, t) \equiv e^{-rt}\left(e^{-\lambda_k t} - e^{-\lambda_i t}\right)$$

For all $\lambda_k < \lambda_i$, $\tilde{f}(\lambda_i, \lambda_k, t)$ is nonnegative, single-peaked in t and reaches a maximum at $t_0(\lambda_i, \lambda_k) = \frac{\ln(\lambda_i + r) - \ln(\lambda_k + r)}{\lambda_i - \lambda_k} > 0$.

Notice also that $\tilde{f}(\lambda_i, \lambda_k, t) = -\tilde{f}(\lambda_k, \lambda_i, t)$, and that

$$\tilde{f}(\lambda_h, \lambda_l, t) = \tilde{f}(\lambda_h, \lambda_m, t) + \tilde{f}(\lambda_m, \lambda_l, t). \tag{25}$$

Finally, $t_0(\lambda_i, \lambda_k)$ is decreasing in λ_i and λ_k : $t_0(\lambda_h, \lambda_m) < t_0(\lambda_h, \lambda_l) < t_0(\lambda_m, \lambda_l)$.

Separating equilibrium

A separating equilibrium (t_l, t_m, t_h) exists if and only the following constraints hold:

$$\begin{cases} W(\lambda_i, \lambda_i, t_i) \geq W(\lambda_i, \lambda_j, t_j) \text{ for all } i \neq j & \text{(26a)} \\ W(\lambda_i, \lambda_i, t_i) \geq W(\lambda_i, \lambda(t), t) \text{ for all } t \notin \{t_l, t_m, t_h\} & \text{(26b)} \end{cases}$$

Let W_i denote the equilibrium payoff of type λ_i.

One may rewrite the set of IC in (26a) as

$$\begin{cases} A\tilde{f}(\lambda_m, \lambda_l, t_l) \leq W_m - W_l \leq A\tilde{f}(\lambda_m, \lambda_l, t_m) & \text{(27a)} \\ A\tilde{f}(\lambda_h, \lambda_l, t_l) \leq W_h - W_l \leq A\tilde{f}(\lambda_h, \lambda_l, t_h) & \text{(27b)} \\ A\tilde{f}(\lambda_h, \lambda_m, t_m) \leq W_h - W_m \leq A\tilde{f}(\lambda_h, \lambda_m, t_h) & \text{(27c)} \end{cases}$$

As usual, in any separating equilibrium, one must have $t_l = t_l^*$. To make things interesting, suppose that

$$\begin{cases} W(\lambda_l, \lambda_m, t_m^*) > W(\lambda_l, \lambda_l, t_l^*) = W_l & \text{(28a)} \\ W(\lambda_l, \lambda_h, t_h^*) > W(\lambda_l, \lambda_l, t_l^*) = W_l & \text{(28b)} \\ W(\lambda_m, \lambda_h, t_h^*) > W(\lambda_m, \lambda_m, t_m^*) = W_m & \text{(28c)} \end{cases}$$

Put differently, if investment were to be efficient for each type, all types would have a

strict incentive to mimic the upper types.

D1 beliefs

Suppose an off-path deviation to t is observed. Let $\Delta_{ij}(t) \equiv W(\lambda_i, \lambda(t), t) - W_i - [W(\lambda_j, \lambda(t), t) - W_j]$ denote the difference between the marginal incentive to deviate to date t for type i and j, when such a deviation generates beliefs $\lambda(t)$.

Using (24), note that $\Delta_{ij}(t) = W_j - W_i + A\tilde{f}(\lambda_i, \lambda_j, t)$, which is independent of $\lambda(t)$.

D1 imposes to prune λ_i if $\Delta_{ij}(t) < 0$.

Specifically,

- we should prune h if $W_h - W_l > A\tilde{f}(\lambda_h, \lambda_l, t)$,
- we should prune h if $W_h - W_m > A\tilde{f}(\lambda_h, \lambda_m, t)$,
- we should prune m if $W_m - W_l > A\tilde{f}(\lambda_m, \lambda_l, t)$

Note that from (25), if one prunes l against m and m against h, then one should prune l against h, so D1 always selects a unique degenerate belief.

Equilibrium analysis

There are four possible configurations:

1. $t_l^* > t_m^* > t_h^* > t_0(\lambda_m, \lambda_l) > t_0(\lambda_h, \lambda_l) > t_0(\lambda_h, \lambda_m)$
2. $t_0(\lambda_m, \lambda_l) > t_m^* > t_l^* > t_h^* > t_0(\lambda_h, \lambda_l) > t_0(\lambda_h, \lambda_m)$
3. $t_0(\lambda_m, \lambda_l) > t_0(\lambda_h, \lambda_l) > t_m^* > t_h^* > t_l^* > t_0(\lambda_h, \lambda_m)$
4. $t_0(\lambda_m, \lambda_l) > t_0(\lambda_h, \lambda_l) > t_0(\lambda_h, \lambda_m) > t_h^* > t_m^* > t_l^*$

Before we characterize the direction of the distortions in each case, let us show the following claims.

Claim 1a: In case 1, one has: $t_0(\lambda_m, \lambda_l) < t_m^*$. This implies $t_0(\lambda_m, \lambda_l) < t_m < t_m^*$.

Proof: Suppose that $t_m < t_0(\lambda_m, \lambda_l) < t_m^*$, and suppose that a deviation to $t \in (t_m, t_0(\lambda_m, \lambda_l))$ is observed.

Since $W_m - W_l \leq A\tilde{f}(\lambda_m, \lambda_l, t_m) \leq A\tilde{f}(\lambda_m, \lambda_l, t)$, D1 imposes to prune λ_l following the deviation to t. This implies $W(\lambda_m, \lambda(t), t) \geq W(\lambda_m, \lambda_m, t) > W(\lambda_m, \lambda_m, t_m)$, so the deviation is profitable to λ_m.

Similarly, one shows that investing at $t_m - \epsilon$ rather than t_m is strictly profitable to λ_m if $t_0(\lambda_m, \lambda_l) < t_m^* < t$. □

Claim 1b: In cases 2-3-4, one has: $t_m^* < t_0(\lambda_m, \lambda_l)$. This implies $t_m^* < t_m < t_0(\lambda_m, \lambda_l)$

Proof: similar as Claim 1a. □

Claim 2a: In cases 1-2-3, one has: $t_0(\lambda_h, \lambda_m) < t_h^*$. This implies $t_0(\lambda_h, \lambda_m) \leq t_h$

Proof: Suppose $t_h < t_0(\lambda_h, \lambda_m) < t_h^*$ and that a deviation to $t \in [t_h, t_0(\lambda_h, \lambda_m)]$ is observed.

Using (27b) and (27c), and $t_0(\lambda_h, \lambda_m) < t_0(\lambda_h, \lambda_l)$, we derive

$W_h - W_l \leq A\tilde{f}(\lambda_h, \lambda_l, t_h) \leq A\tilde{f}(\lambda_h, \lambda_l, t)$, and

$W_h - W_m \leq A\tilde{f}(\lambda_h, \lambda_m, t_h) \leq A\tilde{f}(\lambda_h, \lambda_m, t)$. This implies that the deviation to t should be attributed to λ_h, and is therefore strictly profitable to λ_h. □

Claim 2b: In cases 1-2, one has: $t_0(\lambda_h, \lambda_l) < t_h^*$. This implies $t_h \leq t_h^*$.

Proof: Suppose $t_0(\lambda_h, \lambda_l) < t_h^* < t_h$. and that a deviation to $t \in [t_h^*, t_h]$ is observed

Using (27b) and (27c), we derive

$W_h - W_l \leq A\tilde{f}(\lambda_h, \lambda_l, t_h) \leq A\tilde{f}(\lambda_h, \lambda_l, t)$ and

$W_h - W_m \leq A\tilde{f}(\lambda_h, \lambda_m, t_h) \leq A\tilde{f}(\lambda_h, \lambda_m, t)$. This implies that the deviation to t should be attributed to λ_h, and is therefore strictly profitable to λ_h. □

Claim 2c: In cases 3-4, one has: $t_h^* < t_0(\lambda_h, \lambda_l)$. This implies $t_h \leq t_0(\lambda_h, \lambda_l)$

Proof: Suppose $t_h > t_0(\lambda_h, \lambda_l) > t_0(\lambda_h, \lambda_m)$ and that a deviation to $t \in [t_0(\lambda_h, \lambda_l), t_h]$ is observed.

Using (27b) and (27c), we derive

$W_h - W_l \leq A\tilde{f}(\lambda_h, \lambda_l, t_h) \leq A\tilde{f}(\lambda_h, \lambda_l, t)$ and

$W_h - W_m \leq A\tilde{f}(\lambda_h, \lambda_m, t_h) \leq A\tilde{f}(\lambda_h, \lambda_m, t)$. This implies that the deviation to t should be attributed to λ_h, and is therefore strictly profitable to λ_h. □

Claim 3: In cases 1-2-3, one has $t_m \geq t_h$.

Proof: We know $t_m \geq \min(t_0(\lambda_m, \lambda_l), t_m^*) > t_0(\lambda_h, \lambda_m)$ and $t_h \geq t_0(\lambda_h, \lambda_m)$ (Claim 2a). Therefore, (27c) implies $t_h \leq t_m$. □

Now that we have proven these intermediate results, we can provide necessary conditions in each case.

Case 1 : $t_l^* > t_m^* > t_h^*$

Lemma 6 *In a separating equilibrium, $t_h < t_m < t_l = t_l^*$. In addition, investment is hurried: $t_h < t_h^*$ and $t_m < t_m^*$*

The only inequality which has not yet been proven is $t_h < t_h^*$. To see that it must hold, notice that $t_h < t_m < t_m^*$: Therefore, λ_h can only hurry investment. Investing later than t_h^* would increase the incentive of λ_m to mimic λ_h as compared to the first best, so (27c) cannot hold given that it does not hold in the first best. □

Case 2 : $t_m^* > t_l^* > t_h^*$

Lemma 7 *In a separating equilibrium, $t_h < t_h^* < t_l = t_l^* < t_m^* < t_m$.*

This result is already proven.

Case 3 : $t_m^* > t_h^* > t_l^*$

Lemma 8 *In a separating equilibrium, $t_l = t_l^* < t_h < t_m$. In addition, investment is delayed: $t_h^* < t_h$ and $t_m^* < t_m$.*

Proof: The only inequality which has not yet been proven is $t_h^* < t_h$. To see that it must hold, notice that $t_h \leq t_0(\lambda_h, \lambda_l)$ (Claim 2c). Since $t_l^* \leq t_0(\lambda_h, \lambda_l)$ as well, (27b) implies $t_h \geq t_l^*$. This, together with (28b), implies $t_h > t_h^*$. □

Case 4 : $t_h^* > t_m^* > t_l^*$

Lemma 9 *In a separating equilibrium, investment is delayed: $t_h^* < t_h$ and $t_m^* < t_m$.*

The proof is similar as in Case 3.

9.10 The general case with $r_0 \leq r$

The analysis can be extended to the case where the entrepreneur's initial assets A capitalize at a rate $r_0 \leq r$. The first point to note is that if $r_0 > 0$, there exists a date $\tilde{t}$ after which the entrepreneur no longer needs outside financing, defined by $Ae^{r_0\tilde{t}} = I$.

The expected discounted payoff at date 0 of type $\lambda \in \{\underline{\lambda}, \overline{\lambda}\}$ when he invests at date t, and is perceived as type $\overline{\lambda}$ with probability q now depends on whether the investment date is larger than $\widetilde{t}$ or not. If $t \geq \widetilde{t}$, the entrepreneur does not need outside funds, so asymmetric information has no bite, and

$$W(\lambda, q, t) = W^*(\lambda, t) \text{ for all } (\lambda, q, t).$$

If $t < \widetilde{t}$, (5) becomes

$$x(q,t)p(q,t)R = I - Ae^{r_0 t}.$$

We derive:

$$W(\lambda, q, t) = e^{-rt}s(\lambda, t)\left(p^*(\lambda, t)(1 - x(q,t))R - Ae^{r_0 t}\right).$$

$$\Rightarrow W\left(\overline{\lambda}, q, t\right) - W\left(\underline{\lambda}, q, t\right) = Ae^{-(r-r_0)t} f(t).$$

The function $t \mapsto e^{-(r-r_0)t} f(t)$ is single-peaked and reaches its maximum at

$$t_0 \equiv \frac{\ln(\overline{\lambda} + r - r_0) - \ln(\underline{\lambda} + r - r_0)}{\overline{\lambda} - \underline{\lambda}} > 0.$$

The results of Lemma 1, and of Propositions 2, 3, 4 naturally carry over when appropriately adapting $\tilde{f}$ to:

$$\widetilde{f}(t) = e^{-(r-r_0)t} f(t).$$

We notably derive:

$$\begin{aligned}
A_0 &\equiv \frac{W^*\left(\overline{\lambda}, \overline{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-(r-r_0)\overline{t}^*} f(\overline{t}^*)}, \\
A_1 &\equiv \max\left(0, \frac{W^*\left(\overline{\lambda}, t_0\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-(r-r_0)t_0} f(t_0)}\right), \\
A_2 &\equiv \max\left(0, Ie^{-r_0 t_0} - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0 e^{-(r-r_0)t_0} f(t_0)}\right).
\end{aligned}$$

Accordingly, the equilibrium is unique as well.

The main difference with the case where $r_0 = 0$ is that $\overline{t}^* \geq t_0$ is no longer equivalent to $\overline{t}^* \geq \underline{t}^*$. Therefore, the direction of the distortion is not the one predicted by the ranking of the complete information dates. However, one can show that, for any r_0, there exists a $\hat{p}(r_0)$ such that $\overline{t}^* > t_0$ if and only if $p_0 < \hat{p}(r_0)$. Therefore, our result that projects with a high expected value are delayed, and those with a low expected value are hurried carries over.

Finally, it is easy to see that $\overline{t}^* = \underline{t}^* \Rightarrow \overline{t}^* < t_0$ when $r_0 > 0$ and $\overline{t}^* = \underline{t}^* \Rightarrow \overline{t}^* > t_0$ when $r_0 < 0$. This notably implies that asymmetric information can cause timing reversals. For instance, when $r_0 > 0$, this happens when $\overline{t}^* < \underline{t}^* < \overline{t}$, that is, the good type invests later than the bad type, although he optimally invests earlier under complete information.[25]

[25] A formal proof that this may actually happen is available upon request.