Bayesian DSGE Model Estimation

Lecture two

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Aim of the week

Before getting started - the A, B, C's and D's

The Kalman Filter

Aim of week 2

Know how to derive the VAR model representation of the DSGE model and how to evaluate the Kalman Filter.

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Motivation

A related topic that drew a lot of attention: is there a mapping between the structural shocks of the DSGE model and the innovations of a VAR model? This is important for:

- Model comparison of an estimated DSGE model and a VAR model.
- Estimation of a VAR model.
- DSGE model building to match VAR evidence.
- And also for DSGE model estimation?

A, B C and D

- Attention: In this section notation will deviate from the standard course notation to be in line with the A, B,C's and D's by Fernandez-Villaverde and coauthors.
- Define a state space system as:

$$\breve{x}_t = A\breve{x}_{t-1} + Bv_t \quad (1)$$

$$\breve{y}_t = C\breve{x}_{t-1} + Dv_t \quad (2)$$

- Can the 'true' sequence of shocks be recovered, i.e. does there a exist mapping from the structural shocks of the DSGE model to the innovations of the observable variables?

The VAR model representation

- Solve (2) for v_t and substitute it into (1).
- Given the condition that all eigenvalues of $A - BD^{-1}C$ are strictly smaller than one x_t is given by:

$$\breve{x}_t = \sum_{j=0}^{\infty} [A - BD^{-1}C]^j BD^{-1} \breve{y}_{t-j} \tag{3}$$

- y_t is therefore an VAR model of infinite-order:

$$\breve{y}_t = C \sum_{j=0}^{\infty} [A - BD^{-1}C]^j BD^{-1} \breve{y}_{t-j-1} + Dv_t \tag{4}$$

Finite-order VAR model approximation

(4) can be approximated by the finite-order VAR model:

$$\breve{y}_t = C\sum_{j=0}^{k}[A - BD^{-1}C]^j BD^{-1}\breve{y}_{t-j-1} + Dv_t + \vartheta_{k,t} \tag{5}$$

The approximation error $\vartheta_{k,t}$ is usually assumed to be small.

Aim of the week

Before getting started - the A, B, C’s and D’s

The Kalman Filter

State space system

$$x_t = F(\mu)x_{t-1} + G(\mu)v_t \quad (6)$$

$$X_t = H(\mu)x_t + u_t \quad (7)$$

with:

- $v \sim (0, \Sigma_v)$.
- $u \sim (0, \Sigma_u)$.

The Kalman Filter

- It is an algorithm for calculating linear least squares forecasts of the state vector x on the basis of data observed through date t: $E_{t-1}[x_t] = \hat{x}_t$.
- Kalman Filter calculates these forecasts recursively.
- Associated with the forecast: the mean squared error matrix $\Sigma_{xt} = E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)']$.

$\Rightarrow$ At the end of every iteration step we want an expression for $\hat{x}_{t+1}$ and $\Sigma_{x,t+1}$.

Summary of the Kalman Filter

Given initial conditions for the state vector and its mean square error matrix:

1. Given yesterday's expectation of the state today forecast the observable variables.
2. Compute the corresponding forecast error and update your beliefs about the state today.
3. Given the updated beliefs form expectations about the state tomorrow, their associated forecast error and use those again to forecast the observable variables in 1.

Innovation representation

Define

$$s_t = X_t - \hat{X}_t = X_t - H(\mu)\hat{x}_t \tag{8}$$

Subtracting from (7) yields:

$$s_t = H(\mu)(x_t - \hat{x}_t) + u_t \tag{9}$$

and

$$\Sigma_{st} = H(\mu)\Sigma_{xt}H(\mu)' + \Sigma_u \tag{10}$$

Updating

The optimal forecast of $\hat{x}_{t+1}$ is then given by:

$$\hat{x}_{t+1} = F(\mu)\hat{x}_t + K_t s_t \tag{11}$$

where K_t denotes the Kalman gain and is given by:

$$K_t = F\Sigma_{x,t}H'(H\Sigma_{x,t}H' + \Sigma_u)^{-1} \tag{12}$$

Its corresponding mean square error is updated by:

$$\Sigma_{x,t+1} = F\Sigma_{x,t}F' + G\Sigma_v G' - K_t H\Sigma_{x,t}F' \tag{13}$$

Initial condition

- It is necessary to give an initial condition for x_0 and Σ_{x0} in order to start the Kalman Filter.
- The process for x_t is typically assumed to be covariance stationary.
- The unconditional mean is the solution to: $E[x_{t+1}] = FE[x_t]$.
- The unconditional variance is the solution to the Lyapunov equation: $\Sigma_x = F\Sigma_x F' + G\Sigma_v G'$.
- Thus: $x_0 = 0$, $\Sigma_{x,0} = [I - (F \otimes F)]^{-1} vec(G\Sigma_v G')$

Kalman Filter iteration

The $i-th$ step, given $\hat{x}_{i-1}$, $\Sigma_{x,i-1}$ and X_i. Compute:

1. $s_i = X_i - H(\mu)\hat{x}_{i-1}$.
2. $K_i = F(\mu)\Sigma_{x,i-1}H(\mu)'(H(\mu)\Sigma_{x,i-1}H(\mu)' + \Sigma_u)^{-1}$.
3. $\hat{x}_i = F(\mu)\hat{x}_{i-1} + K_i s_i$
4. $\Sigma_{x,i} = F(\mu)\Sigma_{x,i-1}F(\mu)' + G(\mu)\Sigma_v G(\mu)' -$
   $F(\mu)\Sigma_{x,i-1}H(\mu)'(H(\mu)\Sigma_{x,i-1}H(\mu)' + \Sigma_u)^{-1}H(\mu)\Sigma_{x,i-1}F(\mu)'$
5. Go back to 1.

Convergence results

- Assumption 1: The pair (F, H) is stabilizable.
- Assumption 2: The pair (F', G) is detectable.
- Iterations on the Riccati equation converge: $\Sigma_x \equiv \lim_{t\to\infty} \Sigma_{x,t}$ that satisfies a time invariant equation:
  $\Sigma_x = F\Sigma_x F' + G\Sigma_v G' - F\Sigma_x H'(H\Sigma_x H' + \Sigma_u)^{-1} H\Sigma_x F'$
- Thus there is a time invariant K: $K = F\Sigma_x H'(H\Sigma_x H' + \Sigma_u)^{-1}$.
- The matrix $F - KH$ is a stable matrix.

Derivation of another VAR representation

From

$$\hat{x}_t = [L^{-1} - F]^{-1} K s_t \tag{14}$$

and (8) get

$$X_t = [I + H[L^{-1} - F]^{-1} K L] s_t \tag{15}$$

Use

$$[H[L^{-1} - F]^{-1} K L + I]^{-1} = I - H[I - (F - KH)L]^{-1} K L \tag{16}$$

to get:

$$X_t = H[I - (F - KH)L]^{-1} K X_{t-1} + s_t \tag{17}$$

Another VAR representation

$$X_t = \sum_{j=1}^{\infty} H(F - KH)^{j-1} K X_{t-1} + s_t \tag{18}$$

Homework

1. Write a Matlab code that given a vector of structural parameters:
   - Determines the matrices of the state space system, i.e. solves the DSGE model (use the function from last class).
   - Computes the Kalman Filter iteration steps.
2. Write approximately one page: How would you initialize the Kalman Filter, in case you know that some states are off its stationary distribution, but slowly converging?
3. Not compulsory: A Matlab function that checks whether the DSGE model has a corresponding finite-order VAR model representation.