Time-Consistent Fiscal Guarantee for Monetary Stability*

Gaetano Gaballo
HEC Paris and CEPR

Eric Mengus
HEC Paris and CEPR

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Abstract

Popular theories explaining why fiat money has value rely on some form of commitment to use fiscal power. However, neither monetary stability is a typical fiscal objective nor fiscal commitments may hold at any cost. Thus, to which extent may the pursuit of short-term fiscal objectives sustain long-term monetary stability? This paper provides a theoretical benchmark to answer this question. In a classical OLG monetary model a time-consistent authority maximizes agents' utility and its own expenditures by raising taxes and trading money. Private portfolio choices impose constraints on fiscal plans. Under fairly general conditions, we show that the time-consistent optimal policy selects a unique equilibrium – one in which money is traded efficiently – no matter how small is the weight that the authority attaches to private utility. Uniqueness obtains because the pursuit of short-term consumption equality makes decreasing money holdings privately suboptimal as a by-product. However, when taxing power is limited, money is traded inefficiently and the uniqueness of the equilibrium gets lost if fiscal capacity is not sufficiently large. In particular, a Laffer curve of seigniorage may produce an additional type of equilibrium. In such equilibrium public finances rely less on seigniorage as a higher "inflation tax" reduces private money demand, and so, its "tax base".

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1 Introduction

What gives value to fiat money? Common wisdom has that it rests on the credibility of a public authority who stands ready to sustain it. Several monetary theories dispute, sometimes quite harshly, the exact mechanism through which a certain fiscal intervention can (or cannot) induce people to trade money at a stable value.[1] On the one hand, the Fiscal Theory of the Price Level (FTPL) as developed by Leeper (1991), Woodford (1994, 1995), Sims (1994) and Cochrane (2001, 2005) maintains that money, being a public liability, takes value from the fiscal commitment to future fiscal surplus. On the other hand, Obstfeld and Rogoff (1983)[2] demonstrate that the ability to commit is such a strong power that the presence of a fiscal authority may even be not necessary: anyone endowed with commitment power can use an arbitrarily small redemption value to prevent any form of fiat money – be they shells or cryptocurrencies – to lose value. In sum, the spectrum looks quite large. There is enough to doubt whether monetary stability is an inherent fiscal phenomenon or, on the contrary, can completely abstract from the existence of governments.

Despite such large differences, commitment ability is central in both approaches. This element deserves attention as monetary stability *per se* cannot be considered a typical fiscal objective. A fiscal commitment to preserve monetary stability may actually be too costly to hold when conflicting with more urgent fiscal concerns, like consumption equality or public spending (e.g. because of a sudden health emergency). To which extent may the pursuit of short-term fiscal objectives sustain long-term monetary stability? And how may this affect private incentives to trade money?

This paper laids out the conditions under which a fiscal authority, lacking commitment ability, may sustain monetary stability as a by-product of the pursuit of its short-term fiscal objectives. It provides a theoretical benchmark in the tradition of monetary Overlapping Generations Models (OLG) *à la* Samuelson (1958). In incomplete market models, as OLG are, inflation stability is key to keep consumption equality between lenders and borrowers. Nevertheless, agents may fail to coordinate on such first best allocation because none individually internalizes the social cost of money losing value. To this basic setting we introduce an authority that has the power to tax and carry out money market operations, i.e. it can buy and sell money. This authority implements a time-consistent policy in order to maximize its objectives, which include the utility of present and future agents but also its own expenditures.

Importantly, we assume that the authority is a second mover after private sector's portfolio choices. As in Sims (2013) young agents need to decide whether to save in money holdings or in a storing technology with socially inefficient low return. Storage, in contrast to money, implies a transfer of real resources from one date to the next. A larger amount of storage therefore reduces the availability of resources at a given time, affecting the authority's plans. This feature properly captures the idea that private incentives to acquire money are constraints on fiscal actions as advocated by Bassetto (2002) and Bassetto (2005); we fully embrace his insights and show their relevance in the conduct of the *optimal* fiscal policy.

[1] A vivid example of the state of the debate is condensed in the abstract of Obstfeld and Rogoff (2017).

[2] See also Wallace (1981a) for an earlier discussion of a related argument.

We first establish that, under fairly general conditions, in the baseline case without restrictions on the contingent tax plan, the optimal time consistent policy implements the authority's first best for given private choices. This means that the time consistent policy is robust to any sub-game concern and it is also the one to which an authority would have committed. Moreover, this is also the same policy that a myopic authority, i.e. one only caring about present-time utility, would implement.

We show that this policy selects a unique equilibrium, one where money is efficiently traded, as a by-product of the pursuit of short-term consumption equality, no matter how much it weights in the authority's objective. To get intuition on the mechanism, consider young at time t saving in storage instead of money. The low return on storage leads this generation to consume less than the newly born generation at time $t+1$. Consequently, the resulting inequality will prompt the authority at time $t+1$ to buy money to accrue the real value of money holdings by the old. This means that the return on money from time t to time $t+1$ increases (expected inflation falls) making suboptimal the initial deviation to store. In the extreme case where a generation does not buy money and uses only storage, autarky is prevented because the authority has incentive to exchange any infinitesimal amount of money with a sizeable quantity of consumption goods.

In short, the mere pursuit of short-term consumption equality, no matter how little it weights in the objective of the authority, leads to efficiently fulfill long-run money stability as a consequence. The difference between our mechanism and the one implied by the FTPL goes beyond the time consistent microfoundations of the fiscal intervention. Relating to Sims (2013), we show that in the FTPL the commitment to positive surpluses rules out other equilibria by making *unfeasible* any other equilibrium: with such a commitment, there is simply *no* equilibrium that can form after a deviation. As in Cochrane (2011)'s criticism to Taylor rules, the commitment to positive surpluses corresponds to the commitment to "blow up the economy" for any deviation of the private sector. This contrasts with our time-consistent optimal policy, for which an equilibrium always forms after a deviation by the private sector but, as noted above, in a way that makes the the deviation itself privately suboptimal.

In this respect, our solution relates to Obstfeld and Rogoff (1983) in that an off-equilibrium intervention is essential to stabilize the money market. Nevertheless, the fiscal power of the authority - i.e. the ability to set taxes in response to aggregate savings in storage - is essential to this result. The absence of constraints on taxation implies that there is no trade-off between the authority's own expenditures and its consumption equality objective: by regulating money demand the authority maximizes total consumption, whereas by setting taxes she can keep her own expenditures at the optimal level in any possible state of the world. This is why efficient money trading is achieved irrespective of the weight that the authority gives to agents' utility.

We show that the situation is radically different when the authority is limited in its ability to adjust taxes contingently to private sector's actions, e.g. when taxes are fixed or constrained by a cap. In this case, the authority may face a trade-off between financing its expenditures and money stability. Manipulating the price level is then not only a way to equalize consumption levels across generations but also a way to gather real resources for public expenditures. A unique equilibrium, one in which money is the only saving asset, is still possible if fiscal capacity is high – i.e. if tax revenues are sufficiently high with respect to public spending. However, in this equilibrium, money is traded inefficiently.

When fiscal capacity is low instead the combination of time-consistency and the presence of an alternative to money – storage – leads to the existence of an additional type of equilibrium. In this additional equilibrium, private agents use storage jointly with money, which converges to a positive real value through time. As in a sort of Laffer curve of seigniorage, a higher "inflation tax", that the authority imposes in the attempt to increase its own resources, induces agents to move savings from money to storage. By doing that agents reduce the "tax base" from which the authority drain resources. As a result, the authority ends up in a coordination failure with the private sector in which both consume less compared to the equilibrium where only money is used. In particular, despite in the latter equilibrium primary fiscal deficit is higher, the economy is better off.

Literature review Our paper puts the implicit fiscal guarantee idea vis-à-vis the insights from a popular literature on the interaction between monetary and fiscal policy, as pioneered by Sargent and Wallace (1981). In the same spirit, we study a framework where the conduct of fiscal policy is crucial for monetary stability. In contrast to that stream of literature, in our setting, the presence of a fiscal authority is not only a source of danger; on the contrary, it has an active and essential role in preserving monetary stability. Consistently with Wallace (1981b)'s irrelevance result, we show that interventions require fiscal backing. Yet, this requirement does not imply fiscal interventions in equilibrium but out-of-equilibrium (i.e. a constant zero fiscal surplus does not necessarily rule out the uniqueness of the monetary equilibrium). This feature also distinguishes our theory from the FTPL.

Nicolini (1996) analyzes the mechanism of Obstfeld and Rogoff (1983, 2017) in a model where a fiscal authority decides under discretion the implementation of a costly conversion facility for money. In his model, should hyperinflation occur, there is always a period in which the fixed-cost of the conversion facility will exceed the social costs of hyperinflation. As agents anticipate the stepping in of the authority, hyperinflation does not occur although the facility is not implemented along the equilibrium. In contrast with Nicolini (1996), we assume, as in Sims (2013), that agents face a portfolio choice that effectively constrains the plan of the authority, in the spirit of Bassetto (2002). Absent such feature, our model would always exhibit a unique equilibrium, even in the case of limits to fiscal capacity, consistently with Nicolini (1996).

More recent works about the determination of the price level include Benigno (2017) and Hall and Reis (2016) among others. Although all these works deviate from the typical framework of the fiscal theory of the price level, they are also concerned by the commitment to a particular rule for fiscal transfers without inquiring their optimality and sub-game perfection.

In this respect, we are closer in spirit to Atkeson et al. (2010) and, more generally, to Bassetto (2005), that emphasize how policy implementation is not about committing to unconditional *actions*, but it is about committing to a strategy leading to feasible actions as a function of private agents' decisions. Optimal policy should then make privately suboptimal those actions that are undesired by the authority and that it cannot directly control. In contrast with these papers, we do not assume any form of commitment on the side of the fiscal/monetary authority, consistently with the discussion by Cochrane (2011) on credibility. We share this approach with Barthélemy and Mengus (2018) who investigate how socially expensive is the commitment required to implement a unique equilibrium in typical cashless monetary models.

Other papers investigate the effects of monetary policy rules in overlapping generation models by postulating a demand for money as in Asriyan et al. (2016), to cite a recent example. An other example is Tirole (1985) who considers a situation where the government forces agents to invest some of their savings into a intrinsically worthless asset (that he labels as "gold"). More in general, exogenous motives for money demand obtain by introducing cash-in-advance or money-in-the-utility-function, as reviewed by Walsh (2010). Limit cases of money-in-the-utility-function economies, as defined in Woodford (2003), feature cashless economies like the OLG one studied by Galí (2014). All these approaches rule out equilibria where money loses value by assumption; these equilibria are instead the only source of multiplicity in our paper.

Another related stream of literature models money as one possible emerging medium of exchange in search and matching economies (Kiyotaki and Wright, 1989). Also in those environments one may formalise the idea that the commitment of the government to implement a certain transaction can coordinate agents on the preferred medium of exchange as a unique equilibrium (Aiyagari and Wallace, 1997; Li and Wright, 1998). A natural interpretation of such a commitment is the fact that tax obligations can only be carried out in money as modelled by Starr (1974) and Goldberg (2012) among others. As long as sanctions to tax evasion have negative utility impact, this is another way to enforce real value to money. In any case, Malmberg and Oberg (forthcoming) show that the constraint to pay tax in money is in fact neither a necessary nor a sufficient condition to ensure price level determination.

Our paper is also connected with the literature on multiplicity of equilibria and seignorage revenues as initiated by Bruno and Fischer (1990). In particular, Bruno and Fischer (1990) show that there may exist multiple equilibrium inflation rates consistent with the same seignorage income. In contrast, we show that, in a model where private agents make portfolio choice, there may exist equilibria with higher inflation rates associated with lower seignorage revenues.

In contrast to the typical motive in the literature on bailouts and time-inconsistency (see Schneider and Tornell, 2004; Farhi and Tirole, 2012; Acharya and Yorulmazer, 2007, among others), in our economy, the *ex post* incentive to rescue borrowers is *ex ante* socially desirable. In this context we relate more to Mengus (2017), who shows that government's bailouts in the form of asset purchases can induce intrinsically worthless assets to be traded at positive prices, which may be socially efficient.

2 A Simple Model of Fiat Money

The economy is populated by overlapping generations of homogeneous households of unitary mass and a fiscal authority. Time is discrete and indexed by $t \in \{1, 2, ...\}$. We assume perfect foresight.

2.1 Households

At each date, a new generation of homogeneous agents is born. Each agent lives two periods and then disappears. The representative agent born at time t maximizes the following utility function:

$$U_t \equiv u(C_{y,t}) + \beta u(C_{o,t+1}), \tag{1}$$

where $C_{y,t} \geq 0$ and $C_{y,o} \geq 0$ are individual consumption in the first and second period respectively, $\beta \in (0,1]$ is the discount factor and $u(.) \in \mathcal{U}$ is the utility function. $\mathcal{U}$ denotes a set of multiplicative, continuous and differentiable functions $f(x) : \mathbb{R} \to \mathbb{R}$ with typical concavity properties, i.e. $f'(x) > 0$, $f''(x) < 0$ and $f'(0) \to \infty$. Standard utility functions as CRRA and CARA belong to this family.

Each agent born in period t is endowed with a quantity W_t of consumption good in the first period of his life. The representative young (henceforth "the young") born at time t can submit a quantity of consumption goods $\mathcal{M}_t$ for exchange with fiat money. A market for money, which we will describe in due time, will determine $P_t \geq 0$, i.e. the equilibrium price of consumption in terms of money, and $M_t \geq 0$, i.e. the quantity of money acquired at time t that agents can sell next period once old. A young can also save consumption goods using a storage technology that yields a gross return $\theta \in (0,1)$ in the next period; we denote by $S_t \geq 0$ the amount of goods stored. The budget constraint of the agent born in period t is therefore:

$$C_{y,t} = W_t - T_t - S_t - \mathcal{M}_t, \tag{2}$$

$$C_{o,t+1} = \theta S_t + \frac{M_t}{P_{t+1}}. \tag{3}$$

where T_t denotes a real lump-sum transfer (it can be positive or negative) imposed by the authority on the young.

The first generation is born at date 0, lives just one period, has available a stock of fiat money $M_0 > 0$, does not have storage, $S_{-1} = 0$, and has utility function $U_0 \equiv u(C_{o,1})$. In fact, M_0 denotes a quantity of an asset without intrinsic value whose property rights are assigned to the old of the time-0 generation. This can be interpreted as being central bank's liability, but it does not necessarily need to: for the sake of fixing ideas, one may think about them as be shells or bitcoins.

2.2 The fiscal authority

Let us now introduce a fiscal/monetary authority. We first describe its objectives and then its instruments.

At each date t, the authority maximizes the following objective function:

$$\sum_{\tau \geq t} \beta^{\tau - t} \mathbb{U}_\tau, \tag{4}$$

with

$$\mathbb{U}_\tau \equiv u(C_{y,\tau}) + u(C_{o,\tau}) + \tilde{\lambda} u(G_\tau), \tag{5}$$

being the utility flow at time τ. The authority then cares about the utility of the current representative young and the old agent, but also the level of its own consumption, G_τ weighted by a coefficient $\tilde{\lambda} \geq 0$.

We will say that the case $\tilde{\lambda} = 0$ characterizes one in which the authority is fully benevolent; on the opposite, the case $\tilde{\lambda} \to \infty$ is one in which the authority is fully selfish. However, note that G_τ does not necessarily entail a "waste". The $\tilde{\lambda}u(G_\tau)$ component can be added to the utility of the agents without changing any private consumption choices: in such a case, G_τ would denote a public good whose provision is out of the control of the agents (for example, a public health system).

We denote by $M_{g,t-1}$ the stock of money held by the authority before trading money at time t, with $M_{g,0} = 0$ by convention. The authority can alter its own stock of money by performing money market operations. In particular, it can be either a buyer or a seller of money. In the latter case it will supply a quantity of nominal money that we will denote $M^S_{g,t} \geq 0$, in the former case it submits a quantity of real consumption goods $\mathcal{M}_{g,t} \geq 0$ to be exchanged with money. For technical reasons, we shall assume the constraint $\mathcal{M}_{g,t}M^S_{g,t} = 0$ preventing $\mathcal{M}_{g,t}$ and $M^S_{g,t}$ can be both positive.[3] Before detailing how the money market clears let us rewrite the budget constraint uniquely in terms of the actions that the authority can perform,

$$T_t + \frac{M^S_{g,t}}{P_t} = \mathcal{M}_{g,t} + G_t. \tag{6}$$

In words, the authority can tax or use seigniorage revenues to either consume or buy money from the private sector. We denote by $\Delta_t \equiv \{\mathcal{M}_{g,t}, M^S_{g,t}\}$ the choice of the authority on the money market at time t. We then define a policy of the authority $\mathcal{P}_t \equiv (T_t, \Delta_t, G_t)$ as collection of transfers imposed on the young,[4] money market operations and public consumption that are implemented by the authority at time t.

Assumptions on policy conduct. Given that our primary objective is characterizing the authority's time-consistent policy, we assume that *at date t, the authority cannot commit to future policies $\{\mathcal{P}_\tau\}^\infty_{\tau>t}$*. Moreover, the budget constraint of the authority must hold in any state of the world, i.e. in and off equilibrium. This implies that: i) the equilibrium price in this economy must be determined by restrictions other than (6) and ii) one of the policy variable in $\mathcal{P}$ has to adjust to satisfy (6).

2.3 Timing, market clearing, and equilibrium

Timing. Each time t, actions unfold with the following order:

[3]The only aim of this constraint is technical and without any loss of generality: we want to rule out trivial and uninteresting situations where the authority buys money that itself is supplying, which leaves the allocation unaffected. In fact, what matters to the determination of the value of money is the net position of the authority in the money market.

[4]In Appendix C we provide conditions under which transfers to the old, even if possible, would not be used in equilibrium. Also, as this will become clear, the absence of such an instrument, by itself, will not prevent the authority to implement its first best allocation.

1. The young generation moves first, it fixes the amount of consumption goods to be stored S_t and their real money demand $\mathcal{M}_t$.
2. The authority moves second, it imposes transfers T_t and chooses its position in the money market Δ_t.
3. A market clearing price P_t emerges.
4. The young consumes $C_{y,t}$, the old consumes $C_{o,t}$ and the authority consumes G_t.

Market clearing. We describe here in detail how the market for money works. At the end of the first stage agents have fixed their real money demand $\mathcal{M}_t$. The authority then chooses its position in the money market Δ_t. We have then the following.

Definition 1. For a given available nominal supply of money $M_{t-1} + M_{g,t}^S$ and real money demand $\mathcal{M}_t + \mathcal{M}_{g,t}$ a *market clearing price* P_t is such that

$$P_t(\mathcal{M}_t + \mathcal{M}_{g,t}) = M_{t-1} + M_{g,t}^S \tag{7}$$

where $M_{g,t} = M_{g,t-1} + \mathcal{M}_{g,t}P_t - M_{g,t}^S$ and $M_t = \mathcal{M}_t P_t$ denote the stocks of money held by the authority and the private sector, respectively, after money trade.

It follows that the budget of the authority (6) can be rewritten as:

$$T_t + \frac{M_{g,t-1}}{P_t} = \frac{M_{g,t}}{P_t} + G_t, \tag{8}$$

in terms of the stock of money holdings, and that the total stock of money, $M_t + M_{g,t} = M_{t-1} + M_{g,t-1} = M_0$, stays constant over time. Note that $M_{g,t}$ is left unconstrained; it can be either positive or negative. It is positive when the central bank holds a quantity of the asset initially held by the old generation at time 0 (a fraction of shells or bitcoins in our previous example); it is negative when the central bank issues liabilities which are assimilated to that initial asset. Finally, the markets for goods have to clear at any time, meaning:

$$C_{y,t} + C_{o,t} + G_t = W_t + \theta S_{t-1} - S_t, \tag{9}$$

which imposes that total consumption, including current storage, should equal total available consumption goods, including past storage. Note that S_{t-1} increases the availability of goods at time t.

Authority's control on inflation. Let us denote by $\Pi_{t+1} \equiv P_{t+1}/P_t$ the inflation rate between period t and $t+1$. As the authority is a major actor in the money market and plays after the private sector, it can control the price level, and so inflation, by varying its intervention for given private real money demand. In particular, because of (7) we can state the following.

Lemma 1. *For given P_{t-1} and private real money demand $\mathcal{M}_t$, then*

$$\Pi_t(\Delta_t) = \frac{\mathcal{M}_{t-1} + M^S_{g,t}/P_{t-1}}{\mathcal{M}_t + \mathcal{M}_{g,t}}. \tag{10}$$

entails a one-to-one mapping from Δ_t to Π_t.

By offering more money on the market ($M^S_{g,t} > 0$), the authority can increase the price level and so inflation. In contrast, by demanding money against consumption ($\mathcal{M}_{g,t} > 0$), the authority can decrease the current price level and, thus, reduce inflation. In particular, one can observe that if $M^S_{g,t} = 0$ and $\mathcal{M}_{g,t} > 0$, we have $\Pi_t = 0$ when $M_{t-1} = 0$.

Despite the authority can control inflation, it is a second mover with respect to private agents' portfolio choices. In particular, private storage decisions affect the availability of resources at a given time: the higher the storage the lower real resources available. Therefore, whereas money can transfer consumption from the young to the old, storage transfers consumption across generations. Thus, portfolio choices of agents put effectively constraints on the feasibility of the fiscal plans. This is the essence of the point put forward by Bassetto (2002) (see literature review in the introduction).

Strategies and continuation policies. At each date, a sequence of actions leads to the vector $q_t = \{W_t, S_t, \mathcal{P}_t, M_t, P_t\}$ and, then, to an history h_t defined recursively for $t \geq 1$ by $h_t = \{h_{t-1}, q_t\}$ with $h_0 = \{S_0, M_0\}$ by convention.

For given history h_{t-1} and date-t endowment W_t, a date-t policy function for storage decision and a policy function for real money demands are denoted by $S_t(h_{t-1})$ and $\mathcal{M}_t(h_{t-1})$, respectively. Given history h_{t-1}, endowment W_t, actual storage decision S_t and private real money demand $\mathcal{M}_t$, a date-t policy function for the authority is $\mathcal{P}_t(h_{t-1}, S_t, \mathcal{M}_t)$.

A policy plan is defined as the sequence of policy functions: the policy plan for private agents is $\sigma = \{S_\tau(h_{\tau-1}), \mathcal{M}_\tau(h_{\tau-1})\}_{\tau=1}^\infty$, the policy plan of the government is $\sigma_\mathcal{P} = \{\mathcal{P}_\tau(h_{\tau-1}, S_\tau, \mathcal{M}_t)\}_{\tau=1}^\infty$. Finally, we define $\sigma_t = \{S_t(h_{t-1}), \mathcal{M}_t(h_{t-1})\}$ as agents' policy at time t and $\sigma^t_\mathcal{P} = \{\mathcal{P}_\tau(h_{\tau-1}, S_\tau, \mathcal{M}_t)\}_{\tau=1}^\infty$ as the continuation of $\sigma_\mathcal{P}$ from time t onward.

Equilibrium. For a given initial quantity of money M_0 and sequence of endowments $\{W_\tau\}_{\tau=1}^\infty$, an *equilibrium* is then a set of policy plans $(\sigma^*, \sigma_{\mathcal{P}^*})$ and a sequence of prices $\{P_\tau\}_{\tau=1}^\infty$ such that at any $t \geq 1$,

(i) Given $\sigma^*_\mathcal{P}$, agent's policy σ^*_t maximizes (1) subject to (2)-(3) for any history h_{t-1};

(ii) Given σ^*, the continuation strategy $\sigma^t_{\mathcal{P}^*}$ maximizes (4)-(5) subject to (6) for any history h_{t-1};

(iii) P_t is a market clearing price according to (7).

Note that the definition of equilibrium considers continuations strategies for the authority but only current policies for agents. This occurs because whereas the authority is infinitely-living and so its strategy space concerns an infinite sequence of policies, agents have one-shot choice only, i.e. each generation operates in autonomy from other generations.

3 Optimal portfolios and optimal policy

In this section we determine how the young generation decides real savings and real money demand while anticipating current transfers and future inflation. Then we derive the authority's time-consistent optimal intervention given the private sector's portfolio decisions.

A first important finding is an equivalence result concerning the conduct of the authority's policy. In the absence of constraints on the authority's ability to tax, the authority's time-consistent optimal policy coincides with both the authority's commitment optimal policy and the authority's myopic optimal policy. In other words, the authority's optimal policy is the same irrespective to whether the authority is maximizing its intertemporal objective at time 0 under commitment, or it is just maximizing current utility flow $\mathbb{U}_t$ at each t. In Section 5, we show how the coincidence breaks when the ability to tax is limited.

3.1 Private consumption and portfolio choices

At the beginning of period t, the young generation decides how much to save in real assets and the divide between storage and money holdings. Let us denote by ρ_{t+1} the gross per-unit real return on real savings D_t defined as:

$$D_t \equiv S_t + \mathcal{M}_t.$$

For given $\sigma_{\mathcal{P}}$ the optimal level of real saving D_t is given implicitly by $u'\left(W_t - T_t - D_t\right) = \beta\rho_{t+1}u'\left(\rho_{t+1}D_t\right)$ whereas the split between money and storage is given by arbitrage between the equilibrium return on money Π_{t+1}^{-1} and storage θ. We can then state the following.

Lemma 2 (Optimal private sector policy)**.** *For given sequence of endowments $\{W_\tau\}_{\tau=0}^{\infty}$, history h_{t-1} and authority policy plan $\sigma_{\mathcal{P}}$, the private sector optimal policy $\sigma_t^* \in \sigma^*$ at any date $t \geq 1$ is given by*

$$\begin{aligned}
&S_\tau(h_{t-1}) = 0, \quad \mathcal{M}_t(h_{t-1}) = D_t && \textit{if} \quad \Pi_{t+1}^{-1} > \theta \\
&S_t(h_{t-1}) + \mathcal{M}_t(h_{t-1}) = D_t && \textit{if} \quad \Pi_{t+1}^{-1} = \theta \\
&S_t(h_{t-1}) = D_t, \quad \mathcal{M}_t(h_{t-1}) = 0 && \textit{if} \quad \Pi_{t+1}^{-1} < \theta
\end{aligned}$$

where

$$D_t(h_{t-1}) = \frac{W_t - T_t}{1 + R(\rho_{t+1})}.$$

with $R(\rho_{t+1}) \equiv u'^{,-1}\left(\beta\rho_{t+1}\right)\rho_{t+1}$ and $\rho_{t+1} = \max\{\Pi_{t+1}^{-1}, \theta\}$.

Therefore agents fix their amount of real savings depending on their after-tax income and their preferences. When the return on money is greater (resp. smaller) than the return on real storage, agents save everything in money (resp. storage). Money and storage may coexist only insofar they yield the same return.[5] Thus, saving choices are purely forward looking: the

[5]This conditions can be consistent with randomizations between money and storage. In Appendix D we show that this possibility does not affect our core results, so our focus on symmetric equilibria is without loss of generality.

young generation makes savings' decisions only by looking at future returns. Note that current inflation is not relevant for their current saving decision.

3.2 Optimal policy

Let us start by formally state the problem of the authority.

Problem 1. At any date $t \geq 1$, for given history h_{t-1} and given σ_*, an optimal policy plan by the authority is a $\sigma_{\mathcal{P}*}$ such that any $\sigma^t_{\mathcal{P}*} \in \sigma_{\mathcal{P}*}$ solves:

$$\max_{\sigma^t_{\mathcal{P}}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{U}_\tau,$$

subject to agents' budget constraints (2) and (3), the authority's budget constraint (8), market clearing conditions (7), at any time.

The characterization of the policy function of the authority follows in three key steps.

The first step is noting that the budget constraint of the young can be rewritten independently of current real money demand $\mathcal{M}_t$ and current taxes T_t, as the following lemma states.

Lemma 3. *The level of consumption by the young is given by:*

$$C_{y,t} = W_t - G_t - \Pi_t^{-1}\mathcal{M}_{t-1} - S_t \tag{11}$$

Proof. Use the market clearing condition for money $M_t + M_{g,t} = M_0$ into (8) and solve for T_t. Then substitute T_t into (2) to get (11). □

This is a powerful implication because it shows that the consumption of the young is independent from any return ρ_{t+1}, discount factor β and utility function $u(\cdot)$ but it only depends on storing choices, public consumption and real money holdings of the old.

Given (11), we can rewrite each flow of utility in the authority's objective as:

$$\mathbb{U}_t = u\underbrace{\left(W_t - G_t - \Pi_t^{-1}\mathcal{M}_{t-1} - S_t\right)}_{=C_{y,t}} + u\underbrace{\left(\Pi_t^{-1}\mathcal{M}_{t-1} + \theta S_{t-1}\right)}_{=C_{o,t}} + \tilde{\lambda} u(G_t). \tag{12}$$

Expression (12) reveals the trade-offs at stake in the policy problem. By increasing its real demand for money, the authority deflates the market price as entailed by (10), operating a redistribution of resources from the young in favor of the old. On the other hand, increasing public consumption requires decreasing the consumption of the young.

A crucial observation is that by fixing $(G_t, \Pi_t(\mathcal{M}_t))$ at each t, the authority can influence neither private saving choices $\{S_{t-1}, S_t, \mathcal{M}_{t-1}, \mathcal{M}_t\}$, which the private sector has already fixed, nor future ones which only depend on future inflation rates. Moreover, the ability of the authority to determine the desired inflation rate at each time is unconstrained by their past choices. Therefore, we have the following.

Proposition 4 (Optimal authority policy.). *For given history h_{t-1} and private sector policy plan σ, the authority continuation policy $\sigma^t_{\mathcal{P}^*} \in \sigma_{\mathcal{P}^*}$ at any date $t \geq 1$ is given by $\{\mathcal{P}^*_\tau\}^\infty_{\tau=t}$ such that*

$$\mathcal{P}^*_\tau = \text{argmax}\{\mathbb{U}_\tau\}.$$

The couple $(\sigma^t_{\mathcal{P}^*}, \sigma^*)$ entails the authority's first best allocation; this implies that $\sigma^t_{\mathcal{P}^*}$ is also the continuation policy that the authority would have chosen under commitment at t.

In particular, $\mathcal{P}^*_t = \{\Delta_t, G_t, T_t\}$ is Markovian and given by

- *$\Delta_t(h_{t-1}, S_t, \mathcal{M}_t)$ is such that $C_{y,t} = C_{o,t}$, that is*

$$\Pi_t(h_{t-1}, S_t, \mathcal{M}_t) = \frac{(2+\lambda)\mathcal{M}_{t-1}}{W_t - (1+\lambda)\theta S_{t-1} - S_t} \qquad \textit{if} \quad \lim_{\mathcal{M}_{t-1}\to 0} C_{y,t} \geq \lim_{\mathcal{M}_{t-1}\to 0} C_{o,t} \tag{13}$$

$$\Pi_t(h_{t-1}, S_t, \mathcal{M}_t) \to \infty \qquad \textit{otherwise} \tag{14}$$

according to (10);

- *$G_t(h_{t-1}, S_t, \mathcal{M}_t)$ is such that $G_{y,t} = \lambda C_{o,t}$, that is*

$$G_t(h_{t-1}, S_t, \mathcal{M}_t) = \frac{\lambda}{1+\lambda}(W_t - S_t - \Pi_t^{-1}\mathcal{M}_{t-1}); \tag{15}$$

- *$T_t(h_{t-1}, S_t, \mathcal{M}_t)$ is such that (8) holds, that is*

$$T_t(h_{t-1}, S_t, \mathcal{M}_t) = \frac{1}{1+\lambda}\Pi_t^{-1}\mathcal{M}_{t-1} + \frac{\lambda}{1+\lambda}(W_t - S_t) - \mathcal{M}_t, \tag{16}$$

where we define $\lambda = 1/(u')^{-1}(\tilde{\lambda})$.

Proof. See Appendix B.1 □

The first part of Proposition 4 states that the optimal time-consistent policy is one that maximizes the current flow of utility. This means that a myopic authority, i.e. one that completely neglects the welfare of future generations, would optimally implement the exact same policy of a time-consistent forward looking authority. In addition, the proposition maintains that the same optimal time-consistent policy achieves the authority's first best allocation for given private saving decisions. This implies that the optimal time-consistent policy is also the optimal policy under commitment; more importantly, the authority has never an incentive to deviate from it, no matter what are private sectors' actions. Put simply, the myopic optimal reaction by the authority in the absence of constraints on taxes coincides with the policy decisions of a forward looking authority under commitment. In Section 5, we show how limits on taxes breaks this coincidence by introducing trade-offs between the equality objective – sharing resources between the young and the old generation – and financing the authority's spendings.

The second part of the proposition describes such optimal policy. According to (13), the optimal inflation level is the one that equalizes consumption of the young with the old. To increase the price level, the authority raises real resources by taxing the young generation and uses these resources to purchase money from the old, thus transferring them resources. A corner solution (14) emerges when the young consumes less than the old at the autarky limit, $\mathcal{M}_{t-1} \to 0$, in which case the authority would like to choose a negative money return to transfer resources from the latter to the former; given this is unfeasible, $\Pi_t \to \infty$ emerges. The optimal amount of public consumption (15) is such that the marginal utility of consumption of the young is equal to the marginal utility of public consumption weighted by λ. Taxes (16) clear the budget constraint of the authority.

Remark 1. The relevant parameter in Proposition 4 is $\lambda = 1/(u')^{-1}(\tilde{\lambda})$. In the case where $u = \log$, $\lambda = \tilde{\lambda}$. More generally, λ is an increasing function of $\tilde{\lambda}$ and so that $\lambda = 0$ when $\tilde{\lambda} = 0$ and $\lambda \to \infty$ when $\tilde{\lambda} \to \infty$.

4 Equilibrium

In this section, we characterize the set of equilibria. First, we show that, in the absence of policy intervention, the economy exhibits multiple equilibria because of a complementarity in saving decisions between cohorts. We then show that the implementation of the time-consistent policy leads to a single equilibrium where only money is used. In particular, optimal policy reactions make storage always suboptimal for private agents in comparison with money. This holds true no matter the degree of forward-lookingness of the authority or how much it cares about its own spending (λ) but under the assumption that the authority is not restricted in its ability to tax. We finally provide a detailed discussion on how our mechanism contrasts with the Fiscal Theory of the Price Level.

To ease the presentation, from here onwards, we focus on the case, $W_t = W$ for each t, of a time-invariant endowment, postponing the analysis of the optimal time consistent policy with time-varying endowment to Appendix A.

4.1 Multiplicity in the absence of policy reaction

Let us first describe what is the potential source of multiplicity in our model. To this purpose, let us lay out the equilibrium implied by private sector's optimization in the absence of policy interventions, i.e. with $\mathcal{P}_t = (0, 0, 0)$ at each date t.

In this case, we obtain from combining Lemma 1 and Lemma 2 that equilibrium inflation must satisfy:

$$\Pi_{t+1} = \frac{W/(1 + R(\rho_{t+1})) - S_t}{W/(1 + R(\rho_{t+2})) - S_{t+1}}, \tag{17}$$

given that $M_t = M_0$ for any $t \geq 1$. We can then easily check that, absent policy, a continuum of market equilibria exists as the following proposition states.

Proposition 5. *For any β, a given sequence of endowments $W_t = W$ and policy $\sigma_{\mathcal{P}} = \{0, 0, 0\}$ for any $t \geq 1$, and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple equilibria exist.*

i) *A monetary equilibrium exists such that, at any $t \geq 1$, the private sector policy $\sigma_t^* \in \sigma^*$ is* *given by*

$$S_t = 0, \quad \mathcal{M}_t = \frac{W}{1+R(1)} \quad \textit{and} \quad \Pi_{t+1} = 1 \tag{18}$$

with $P_t = P^*$ *with* $P^* \equiv 2M_0/W$.

ii) *An asymptotic autarky equilibrium exists for each* $s \geq 1$ *such that, the private sector policy* $\sigma_t^* \in \sigma^*$ *is given by* (18) *for* $t \leq s$*, and by*

$$S_t = \frac{W}{1+R(\theta)} - \theta\mathcal{M}_t, \quad \mathcal{M}_{t+1} = \theta\mathcal{M}_t \quad \textit{and} \quad \Pi_{t+1} = \theta^{-1}$$

for $t > s$*, with* $P_s \in (P^*, \theta^{-1}P^*)$ *and* $\mathcal{M}_s = M_0/P_s$.

iii) *An autarky equilibrium exists where the private sector policy* $\sigma_t^* \in \sigma^*$ *at any date* $t \geq 1$ *is given by* $\mathcal{M}_t = 0$, $S_t = W/(1+R(\theta))$ *and* $P_t \to \infty$.

Proof. See Appendix B.2 □

We illustrate the multiplicity of equilibria in Figure 1, where we assume $\beta = 1$, $\theta = 0.95$ and $W = 0.3$. The source of this multiplicity is the complementarity of storage decisions across generations. Suppose the generation born at time t decides to increase their storage. This reduces their real money demand, i.e. the amount of consumption they are willing to exchange for money. The return on money from period $t-1$ to t has then to decline. In turn, storage becomes a relatively more profitable investment for the generation born at time $t-1$. In particular, as storage at time t becomes sufficiently large to lower the return on money strictly below θ then it is optimal for the generation born at time $t-1$ to also store goods. In the end, this complementarity in portfolio decisions leads to the following equilibria.

A *monetary equilibrium* exists where agents never use storage. Agents then perfectly equalize consumption across periods. This equilibrium, which is denoted with a circle marker in Figure 1, is characterized by a constant real demand for money $\mathcal{M}_t = W/(1+R(1))$, constant prices $\Pi_t = 1$ and no storage.

In addition to this equilibrium, there also exist equilibria where storage and money are both used and storage progressively crowds out monetary savings. We call this kind of equilibria *asymptotic autarky equilibria*. As storage and money are used at the same time, in these equilibria $\Pi_t = \theta^{-1}$ holds since arbitrage between the two saving assets must no be possible. Along these paths real money demand follows the process:

$$\mathcal{M}_{t+1} = \theta\mathcal{M}_t, \tag{19}$$

i.e. lower real money demand today depresses future real money demand, so that storage crowds out money as time goes on. In the end, storage converges to $\lim_{t\to\infty} S_t = W/(1+R(\theta))$. Given that $M_0/P_t = \mathcal{M}_t$ in the absence of intervention and $\mathcal{M}_t$ converges to 0, money has ultimately no real value, i.e. $\lim_{t\to\infty} M_0/P_t = 0$. These equilibria are denoted with a cross marker in Figure 1. Importantly, notice that storage can jump in any period from zero to positive since

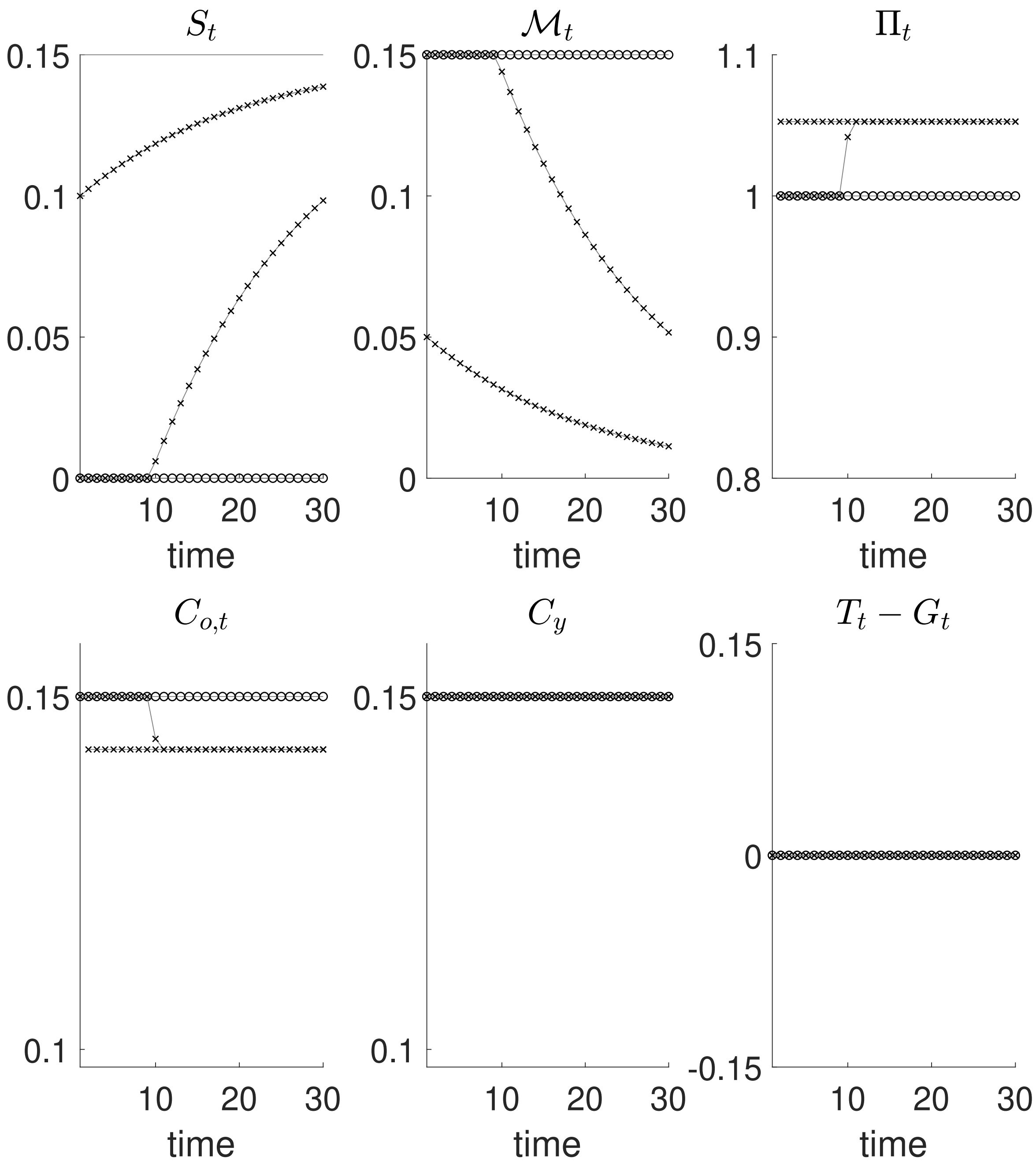

Figure 1 – Equilibria with no policy interventions for $\beta = 1$, $\theta = 0.95$, $W = 0.3, \lambda = 0$. Circles denote the monetary equilibrium, cross markers denote two asymptotic autarky equilibria: one that starts at $S_0 = 0.1$ the other at $S_0 = 0$ with a jump at $S_{10} = 0.006$. Autarky, that is possible in this case, is denoted by a solid line.

there are positive levels of S_t compatible with $\Pi_t < \theta^{-1}$ for which $S_{t-1} = 0$ is optimal. In the Figure we provide an example where storage jumps to a positive value at $S_{11} = 0.027$ after ten periods. However, for S_t to be positive then $\Pi_{t+1} = \theta^{-1}$, which implies $S_{t+1} > S_t$. So storage can jump from zero to positive at any period, but only once along any asymptotic autarky equilibrium.

An *autarky equilibrium* exists in the absence of policy interventions. It is represented with a single solid line in Figure 1. In this case, storage is maximal, the real value of monetary savings is zero, with prices being infinitely large (so that inflation is not defined). Consumption profiles are the same as in an asymptotic autarky equilibrium with storage and money as the return to savings is the same.

4.2 Equilibrium with optimal policy intervention

Let us turn now to the case where policy interventions are optimally chosen as determined by Proposition 4. We first provide a set of equations characterizing the equilibrium outcome, then we describe the equilibrium set. This set boils down to the only monetary equilibrium. We finally provide intuition on why policy interventions lead to a single equilibrium.

Equilibrium characterization To start with, let us focus on the equilibrium conditions implied by the private sector. First, by combining the young generation budget constraint with the optimal level of taxes set by the authority, we are able to compute the real demand for money at date t:

$$\mathcal{M}_t = \frac{W - (1 + R(\rho_{t+1})(2+\lambda))S_t + \theta S_{t-1}}{(2+\lambda)R(\rho_{t+1})} \tag{20}$$

Using (13) at date $t+1$, we can recover the actual law of motion for inflation at any state as:

$$\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t} = \frac{1}{R(\rho_{t+1})} \frac{W - (1 + R(\rho_{t+1})(2+\lambda))S_t + \theta S_{t-1}}{W - (1+\lambda)\theta S_t - S_{t+1}}, \tag{21}$$

which must always hold. It is easy to check that for $S_t = 0$ at any t inflation is equal to $\Pi_{t+1} = R^{-1}(\Pi_{t+1}^{-1}) = \beta$.

The equilibrium set Now let us investigate the equilibrium set once policy is optimally set. Formally, this requires the equilibrium allocation to satisfy (21). As shown by the following proposition, this set of equilibria boils down to the monetary equilibrium:

Proposition 6. *For any $\{\lambda, \beta\}$, a given sequence of endowments $W_t = W$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, a unique equilibrium exists, one in which money is efficiently traded. In such equilibrium, at any $t \geq 1$:*

(i) $\sigma_t^* \in \sigma^*$ is such that:

$$S_t = 0 \quad \textit{and} \quad \mathcal{M}_t = \beta \frac{W}{2+\lambda},$$

(ii) $\sigma^t_{\mathcal{P}^*} \in \sigma_{\mathcal{P}^*}$ *is such that:*

$$\begin{aligned}
\Pi_t &= 1/R\left(\Pi_t^{-1}\right) = \beta \\
G_t &= \frac{\lambda}{2+\lambda} W \\
T_t &= \frac{1+\lambda-\beta}{2+\lambda} W,
\end{aligned}$$

(iii) the price level is given by $P_t = M_t/\mathcal{M}_t$ *where* $M_t = \beta M_{t-1}$.

Furthermore, for any given $S_t \in (0, W/(1+R(\theta))]$*, a unique equilibrium exists; in this equilibrium, storage monotonically shrinks reaching the steady state described above.*

Proof. See Appendix B.3 □

The proposition states that the implementation of optimal time-consistent policy interventions implies a unique equilibrium for any given level of storage. For the initial condition $S_0 = 0$, the unique equilibrium is one in which agents use money as the only saving asset. This equilibrium, differently from the pure monetary equilibrium described above, is characterized by an inflation rate equal to the inverse of the discount factor of agents. This is indeed a socially efficient outcome. To achieve this result, the authority taxes the young generation to buy money at a fixed rate. In particular, both private money holdings and prices shrink at a rate β consistently with a fixed real money demand.

The last part of the proposition states that there exists a unique continuation equilibrium for any *given* positive S_t. This is particularly important to evaluate the outcomes of a deviation from the unique equilibrium, i.e. a jump to a positive S_t. The existence of a continuation value for any state of the economy is an important feature of our theory in contrast to the FTPL; we will discuss at length in Section 4.3. In Corollary A.2, stated in Appendix, we show the closed-form solution of such continuation equilibrium, obtained as a particular case of the one for time-varying endowments (Proposition A.1).

No commitment technology is required then to sustain monetary stability to the extend that, as discussed above, even a not-fully-benevolent and myopic government would optimally implement the same optimal policy. Related to this, it is important to note that the proposition is independent of the weight on authority's expenditures λ. In other words, no matter how benevolent or forward looking the authority is, the unique equilibrium provides for the real value of money being at the socially efficient level. The key intuition for the irrelevance of λ is that the consumption of the government is a fraction of the consumption of the old (which, in this equilibrium, is equal to one of the young), who is better off in an economy where money has value. As a result, whatever the value of λ, the authority always prefers the economy to stay in the monetary equilibrium where everyone, the authority itself included, is better off.[6]

In particular, the equilibrium is unique also in the case $\beta = 1$, in which the primary surplus is zero. These features point out to a sharp difference with the Fiscal Theory of Price Level, which

[6] This statement generalizes to the case that the authority gives a sufficiently large relative weight to money holders, i.e. the old generation.

we will discuss in 4.3. Before that let us explain the source of the uniqueness of equilibrium.

How policy interventions rule out multiple equilibria? The key reason why there exists only one monetary equilibrium is that, due to out-of-equilibrium policy interventions, private agents have no incentives to deviate and use storage.

We illustrate this logic in Figure 2. In this Figure, we plot the pure monetary equilibrium with a circles, but also the continuation of an equilibrium for a *given* S_t with cross markers. The figure is produced with the same parametrization of Figure 1 except that we now assume a $\lambda = 0.5$. Note that, along the pure monetary equilibrium, because the authority cares about its own consumption, private consumption is lower than in Figure 1 as taxes are raised. On the other hand, the case $\beta = 1$ plotted in the figure, corresponds to a monetary equilibrium where inflation is equal to one and primary fiscal surplus is zero, i.e. public spending is completely financed by taxes.

In analogy with Figure 1, we explore a potential equilibrium starting at $S_0 = 0$ with a jump to positive storage at $S_{10} = 0.006$. The dotted line with cross markers denotes the ideal path of storage that would have sustained such a move. In particular, in analogy with the reasoning in absence of policy, positive storage at time $t = 10$ could be sustained only by a belief of higher storage at time $t = 11$ and so on. However, as the picture shows, for a *given* positive level of storage at time $t = 11$ there exists a unique equilibrium denoted by a solid dark line with cross markers where storage decreases at time $t = 12$ before converging to the monetary equilibrium. Crucially, this implies, according to (21), an inflation rate Π_{12} from period $t = 12$ to $t = 13$ drops much lower than $\theta^{-1} = 1/0.95$, producing a return on money strictly higher than the one on storage. For that rate of inflation the young at $t = 11$ would have never optimally chosen to store any unit in storage! By anticipating that no individual would then rationally choose $S_{11} > 0$, no jump to positive storage at $S_{11} > 0$ can occur.

Let us now describe how the fiscal incentives of the authority shape the paths just described. The objective of the authority of financing its own consumption is completely covered by taxes, primary surplus results therefore from the implementation of market operations. As Figure 2 shows in any path in- and out-of-equilibrium consumption equality between young and old is endured by the policy. In particular, consider the out-of-equilibrium dotted path denoted by cross markers. The increase in storage by the young reduces the real value of private money demand and so generates downward pressure on money return. In this case, the authority reacts by taxing the young to buy money ($\mathcal{M}_{g,t} > 0$ and $M_t^S = 0$) in order to sustain its value, and by doing that, the consumption of the old. By Lemma 1, for $\Pi_t = \theta^{-1}$ we get the analogous to (17) with optimal policy interventions:

$$T_t - G_t + \mathcal{M}_t = \theta \mathcal{M}_{t-1} \tag{22}$$

where the additional term captures the intervention. In particular, the optimal real surplus

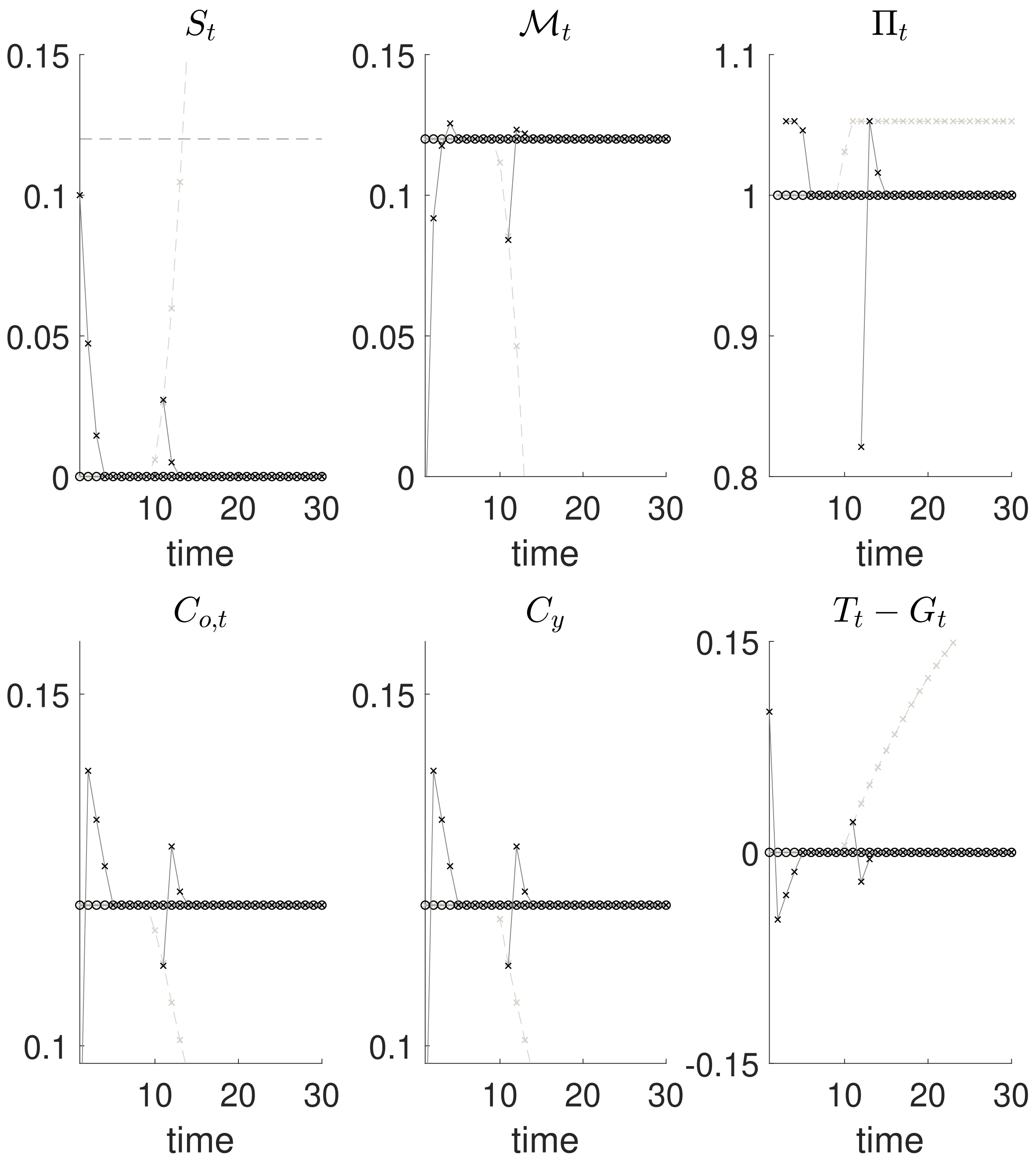

Figure 2 – Equilibria with optimal policy for $\beta = 1$, $\theta = 0.95$, $W = 0.3, \lambda = 0.5$. Circles denote the pure monetary equilibrium, cross markers denote two equilibria with storage: one that starts at $S_0 = 0.1$ the other at $S_{11} = 0.027$. We also denote with a light grey dashed line the unfeasible path of an equilibrium that starts at $S_0 = 0$ with a jump at $S_{10} = 0.006$, which requires $S_{11} = 0.027$. Autarky, that is not possible in this case, is denoted by a simple dashed line.

decided by the authority in response to past storage choices evolves according to

$$T_t = \left(1 + \lambda - R_t^{-1}\right) \frac{W}{2+\lambda} + (1 + R_t^{-1}) \left(\frac{S_t - \theta S_{t-1}}{2+\lambda}\right) \tag{23}$$

$$G_t = \frac{\lambda}{2+\lambda} \left(W - S_t + \theta S_{t-1}\right). \tag{24}$$

Thus, increasing storage goes along with increasing primary surplus and decreasing (but equally split) consumption.

On the contrary, along a path where storage decreases – any in-equilibrium dark solid path denoted by cross markers – the authority implements a negative surplus: it sells out money in its balance sheet and transfers seigniorage revenues to the young. To understand the optimality of this behaviour, note that a *given* amount of positive storage at time t increases the availability of resources available at time $t+1$. As said, storage in contrast to money transfers resources across periods, so that there is an optimal decrease of storage that balances the utility of transferring resources to the next generation and the depreciation cost of waiting one more period before consuming. This path is implemented even by a myopic authority. In fact, as the authority ensures the efficient split of resources between old and young of two subsequent cohort at the same period, private sector choices ensure the efficient split of resources between young and old within the same cohort across two subsequent periods, so that the optimal allocation is achieved.

Let us conclude by discussing the autarky situation where a generation decides to save only using storage. This is not an equilibrium either with policy intervention. As we show in the proof of Proposition 6, in an autarky situation, the authority at time t has an incentive to put real resources in exchange from the money bought by young at time $t-1$ (as a deviation from autarky), no matter how small the deviation is. This leads to an infinite return on money. To see this suppose an individual young at time $t-1$ buys an arbitrarily small but strictly positive amount of money, whereas none else neither in her cohort or in the next cohort does, i.e. $\mathcal{M}_{t-1} = \epsilon$ and $\mathcal{M}_t = 0$, with $\epsilon > 0$ but arbitrarily small. According to (10), any combination $\mathcal{M}_{g,t} > 0$ and $M^S_{g,t} = 0$, leads to $\Pi \to 0$ and then to an infinite return to money. Because of the profitability of any individual deviation from autarky, autarky cannot be an equilibrium.

4.3 Comparison with the Fiscal Theory of the Price Level

Following the Fiscal Theory of the Price Level, Sims (2013) shows that committing to a sequence of positive surpluses leads to a single equilibrium. Having built our model on the basic structure of Sims (2013) makes very transparent the contrast of our theory with the logic of the Fiscal Theory of the Price Level. The main difference is the following. Whereas our optimal time-consistent policy leads to uniqueness because it makes certain private actions suboptimal, the Fiscal Theory of the Price Level rules out potential equilibria making them unfeasible, i.e. preventing that *any equilibrium path* can form after a deviation by the private sector. Let us see this in details.

Commitment to primary fiscal surplus. In our model, we can fix the sequence of surpluses to positive by restricting the set of policies that we consider: $\mathcal{P}_t = \left\{\bar{T}, \Delta_t, \bar{G}\right\}$ where $\bar{T} - \bar{G} > 0$, with $\bar{T} > 0$ and $\bar{G} \geq 0$.

Let us first note that under this restriction, Δ_t is constrained: the budget constraint at time

t is:

$$\bar{T} - \bar{G} = \mathcal{M}_{g,t} - \frac{M_t^S}{P_t}.$$

As a result, when $\bar{T} - \bar{G} > 0$, $M_t^S = 0$ and $\mathcal{M}_{g,t} = \bar{T} - \bar{G}$.[7] The market clearing condition for money then implies that:

$$\mathcal{M}_t = \mathcal{M}_{t-1}\Pi_t^{-1} - \bar{T} + \bar{G} \tag{25}$$

Let us now turn to the equilibrium outcome when the sequence of surpluses is fixed. To start with, let us note that, when the inflation rate Π_t is always weakly larger than 1, one can observe that $\mathcal{M}_t$ ultimately becomes negative. This happens also in the case where $\Pi_t = 1$ at all dates. However, $\mathcal{M}_t$ cannot be negative and so there cannot be an equilibrium where $\Pi_t > 1$ at any date t. This then implies that, in equilibrium, Π_t cannot be always larger than 1: it has to be, at least sometimes, smaller than 1.

In particular, as noted by Sims (2013), an equilibrium where storage is used and where $\Pi_t = \theta^{-1}$ is not possible. If agents select a positive storage $S_t > 0$, there is simply no feasible paths consistent with permanent positive surpluses.

The following proposition shows that there is in fact only one equilibrium consistent with fixed surpluses:

Proposition 7. *Suppose that $u(\cdot) = \log(\cdot)$. When the authority commits to a sequence of strictly positive surpluses ($\bar{T} - \bar{G} > 0$), there exists a single equilibrium where the inflation rate is, at any date t:*

$$\Pi_t = \bar{\Pi} \equiv \frac{W - \bar{T}}{W - \bar{T} + 2(\bar{T} - \bar{G})} < 1. \tag{26}$$

Proof. See Appendix B.4 □

Committing to a fixed sequence of positive surpluses then leads to a single equilibrium, at least with log utility (see final remark).

However, this result is deeply different compared with Proposition 6 for two reasons.

First, fixing the sequence of surpluses leads to intervene not only *off-equilibrium* but also *in equilibrium*: the authority has to run surpluses that lead to deflation. This equilibrium deflation is in general suboptimal, except when taxes and expenditures are fixed so that:

$$\frac{W - \bar{T}}{W - \bar{T} + 2(\bar{T} - \bar{G})} = \beta < 1.$$

In particular, it is always suboptimal when $\beta = 1$ as this would require to set surpluses to $\bar{T} - \bar{G} = 0$. In contrast, Proposition 6 does not require strictly positive surpluses in equilibrium

[7] One may wonder whether this does not stem from our assumption that the authority can either buy or sell money but not both at the same time. However, as we explained, relaxing this assumption would only allow the authority to issue money that it itself buys, without changing net purchases of money.

for a unique equilibrium to exists – this can be observed in the case where $\beta = 1$ – but only to interventions out of equilibrium. To borrow Bassetto (2005), this is because Sims (2013) implies to commit to *actions* while the optimal policy as considered in Proposition 6 is a *strategy*, where actions by the authority are functions of actions of private sector.

Second, and more importantly, the reason why there does not exist another equilibrium is different: committing to fixed surpluses rules, i.e. always buy money, out alternative equilibria because no equilibrium can form after a deviation by the private sector such as a decision to store $S_t > 0$.

The proof is informative about how such a commitment makes unfeasible positive storage. Using equation (26), we show that, if at some point agents store, then the commitment to always buy the same real amount of money leads the government to buy the entire stock of money within a finite number of periods. In the next period, there is no money left in the hands of private agents but the government has still to buy some money, which is not possible. Said differently, in the case of a deviation by the private sector, the commitment to positive surpluses by the private sector amounts to a commitment to push the economy to diverge to violate a feasibility condition. In analogy to Cochrane (2011)'s critique of Taylor rules, such a commitment to push this economy to diverge in case of a non-desired private action would correspond to "blow up the economy".[8]

Such a commitment is then time-inconsistent: after a deviation, the authority might prefer to adjust spending and taxes so that an equilibrium can form. This is actually what happens with the optimal time-consistent policy.

This then leads to a key difference between Sims (2013) – and, more generally, with the FTPL – and our main result. In our theory, optimal policy interventions preclude alternative equilibria in a time-consistent way: it is not optimal for the private sector to deviate. Conditional on a deviation by the private sector, there does exist a continuation equilibrium. However, this continuation equilibrium implies that the deviation (i.e. using storage) is not optimal from an ex-ante point of view.

Remark 2. Note that Proposition 7 extends to CRRA preferences where the intertemporal elasticity of substitution is higher than 1. The important ingredient is that lower money holdings is either consistent with storage or with higher inflation. This is not necessarily satisfied when the intertemporal elasticity of substitution is lower than: in this case a higher inflation rate today corresponds to a lower real money demand today, which may be consistent with a lower inflation rate tomorrow. As a result, there may still be multiple equilibrium paths for inflation in this case.

[8]The exact words from Cochrane (2011) are "A policy configuration for which "no equilibrium can form" or "private first-order conditions cannot hold" means a threat to blow up the economy." He uses these words to describe, among others, the logic of the Taylor rule, which following his description, corresponds to the commitment to push inflation to diverge except for the desired level of inflation.

5 Limits to guarantees

Equilibrium uniqueness is obtained in Proposition 6 under the assumption that the authority can set taxes contingently on private agents' actions without any constraint (except feasibility). In this section, we discuss how limitations to the authority's contingent tax plan may hamper the uniqueness of the equilibrium and eventually change the set of equilibria. We consider situations where the authority either cannot modify taxes or there is a cap on its fiscal capacity. We show that when the authority is not sufficiently benevolent or when the constraints on ability to tax is too tight, monetary stability gets lost and multiple equilibria arise.

The main reason is that the inability to adjust taxes leads to a trade-off between monetary stability and the authority's own expenditures. When the authority needs more resources for its own consumption, it cannot resist raising seigniorage, which harms money stability. Importantly, in this setting the authority still controls the price level, but it can end up in a coordination failure with agents; a form of Laffer curve of seignorage exists. Key to this result is the timing of the portfolio choice of agents: the authority plays after private saving choices, which affect the availability of resources. In particular, in the attempt to finance higher public spending, the authority produces a higher inflation rate. A higher inflation rate incentives the reallocation of private savings from money holdings to storage, reducing total amount of available resources. In such a situation both actual seignorage revenues and private consumption decrease.

5.1 Fixed taxes

We first explore the equilibrium outcome when the authority cannot change taxes in reaction to saving choices. This situation can be interpreted as one in which the authority commits ex-ante to a given fiscal stance as well as one in which the authority is a private entity lacking therefore any fiscal power (i.e. ability to generate revenues at will). We show that such a situation leads to a trade-off for the authority between intervening in the money market and financing its own expenditures. When the level of taxes is sufficiently high, the uniqueness of the monetary equilibrium is possible, but money is traded inefficiently. When instead the level of taxes is too low, the authority favors its expenditures over purchases of money, thus giving rise to multiple equilibria. In particular, a new equilibrium is possible in which money and storage both coexists, but money never loses value.

Optimal time-consistent policy. Fixing taxes amounts to restricting the policies space to $\hat{\mathcal{P}}_t = (\Delta_t, G_t, \bar{T})$ where, to maintain the analogy with the benchmark model, taxes on the young $T_t = \bar{T}$ are taken fixed through time. In contrast with the previous subsection, the authority can only back its interventions in the money market by adjusting its expenditures.

As in the benchmark case, the first step is to rewrite the consumption of the young independently of date-t variables:

$$C_{y,t} = W - \bar{T} - S_t - \mathcal{M}_t = \frac{R(\rho_{t+1})(W_t - \bar{T})}{1 + R(\rho_{t+1})}. \tag{27}$$

and real saving then writes as $S_t + \mathcal{M}_t = C_{y,t}/R(\rho_{t+1})$. By combining the authority's budget constraint with the saving equation $S_t + \mathcal{M}_t = C_{y,t}/R(\rho_{t+1})$, we obtain spending G_t as follows:

$$G_t = \frac{W_t + R(\rho_{t+1})\bar{T}}{1 + R(\rho_{t+1})} - S_t - \Pi_t^{-1}\mathcal{M}_{t-1}. \tag{28}$$

This equation clarifies that, in case of need of additional resources, the authority has no other option than draining them via higher inflation as the other terms are policy independent. Importantly, this ability to raise resources through inflation depends on the real money holdings of the previous period $\mathcal{M}_{t-1}$: the less the old generation has invested in money (and the more it has invested in storage), the more the authority has to raise inflation to finance its expenditures. There is then a complementarity between the use of storage by agents and the incentive to raise inflation by the authority in the next period. Such complementarity does not show up when taxes can be adjusted.

Consumption and portfolio choices are still as described in section 3.1. Instead, optimal policy is different. In analogy with the first part of Proposition 4, the optimal time-consistent policy is given by $\hat{\mathcal{P}}_t^* = \text{argmax}\{\mathbb{U}_t\}$, i.e. the one that, at each time t, maximizes the current flow of the authority's utility. In contrast with the previous case, the current flow of utility in the authority's objective is now given by:

$$\mathbb{U}_t = u\left(C_{y,t}\right) + u\underbrace{\left(\Pi_t^{-1}\mathcal{M}_{t-1} + \theta S_{t-1}\right)}_{=C_{o,t}} + \tilde{\lambda}u\underbrace{\left(\frac{W_t + R(\rho_{t+1})\bar{T}}{1 + R(\rho_{t+1})} - S_t - \mathcal{M}_{t-1}\Pi_t^{-1}\right)}_{=G_t}, \tag{29}$$

where $C_{y,t}$, according to (27), is independent from policy. The solution to this problem, $\hat{\mathcal{P}}_t^* = \{\Delta_t, G_t, \bar{T}\}$ is Markovian and given by

- $\Delta_t(h_{t-1}, S_t, \mathcal{M}_t)$ is such that

$$\Pi_t = \frac{(1+\lambda)\mathcal{M}_{t-1}}{\frac{W_t + R(\rho_{t+1})\bar{T}}{1+R(\rho_{t+1})} - \lambda\theta S_{t-1} - S_t} \quad if \lim_{\mathcal{M}_{t-1}\to\infty} \lambda C_{o,t} \leq \lim_{\mathcal{M}_{t-1}\to\infty} G_t, \tag{30}$$

$$\Pi_t \to \infty \quad \text{otherwise,} \tag{31}$$

  according to (10);

- $G_t(h_{t-1}, S_t, \mathcal{M}_t)$ given by (28);

at any t, where, again, $\lambda = 1/(u')^{-1}(\tilde{\lambda})$. By losing the ability to change taxes in response to private saving choices the authority loses the ability to influence the demand of savings and so the consumption of the young. There is now a trade-off in the use of the price for money as an instrument. On the one hand, the authority may reduce consumption inequality by lowering the price for money. On the other hand, it can increase public expenditures by decreasing the price for money. Which force prevails depends on the initial level and the importance of public expenditures.

Equilibrium. Let us now turn to the equilibrium outcome when policies solve (30). In contrast to Section 4, where taxes are unconstrained, we obtain here a multiplicity of equilibria depending on the level of taxes. The following proposition describes these findings.

Proposition 8. *For any β, a given sequence of endowments $W_t = W$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple equilibria exist depending on $\{\bar{T}, \lambda\}$.*

(a) *Provided that*

$$\hat{\pi} \equiv (1+\lambda)\frac{W-\bar{T}}{W+\bar{T}} \leq \theta^{-1},$$

an inefficient monetary equilibrium exists such that, for any $t \geq 1$:

(i) $\sigma_t^* \in \sigma^*$ is such that:

$$S_t = 0 \quad \text{and} \quad \mathcal{M}_t = \frac{W-\bar{T}}{1+R(\Pi^{-1})}, \tag{32}$$

(ii) $\sigma_{\hat{\mathcal{P}}*}^t \in \sigma_{\hat{\mathcal{P}}*}$ is such that:

$$\Pi_t = \hat{\pi}, \tag{33}$$

$$G_t = \frac{(1+\lambda)(W+R(\hat{\pi}^{-1})\bar{T}) - (W+\bar{T})}{(1+R(\hat{\pi}^{-1}))(1+\lambda)}, \tag{34}$$

(iii) the price level is given by $P_t = M_t/\mathcal{M}_t$ where $M_t = \hat{\pi} M_{t-1}$.

(b) *Furthermore, when*

$$\frac{\bar{T}}{W} < \frac{\lambda\theta}{1+R(\theta)+\lambda\theta},$$

- *a money-storage equilibrium exists for each $s \geq 1$ such that, (32)-(34) holds for $t < s$, and for any $t \geq s$:*

  (i) $\sigma_t^* \in \sigma^*$ is such that:

$$S_t = \theta S_{t-1} + \frac{W+R(\theta)\bar{T} - \theta(1+\lambda)(W-\bar{T})}{1+R(\theta)},$$

$$\mathcal{M}_t = \frac{W-\bar{T}}{1+R(\theta)} - S_t \quad \text{with} \quad \lim_{t\to\infty} \mathcal{M}_t = \frac{\theta\lambda(W-\bar{T}) + (R(\theta)-1)\bar{T}}{(1-\theta)(1+R(\theta))} \geq 0,$$

  (ii) $\sigma_{\hat{\mathcal{P}}*}^t \in \sigma_{\hat{\mathcal{P}}*}$ is such that:

$$\Pi_t = \theta^{-1} = \frac{1}{R(\theta)},$$

$$G_t = \lambda\theta\frac{W-\bar{T}}{1+R(\theta)},$$

  (iii) the price level is given by $P_t = M_t/\mathcal{M}_t$ where $M_t = \theta^{-1} M_{t-1}$.

- *and an autarky equilibrium exists where $\sigma_t^* \in \sigma^*$ is such that $S_t = (W-\bar{T})/(1+R(\theta))$, $\mathcal{M}_t = 0$, and $\sigma_{\hat{\mathcal{P}}*}^t \in \sigma_{\hat{\mathcal{P}}*}$ such that $\Pi_t > \theta^{-1}$, $G_t = \bar{T}$ and $P_t \to \infty$, for any $t \geq 1$.*

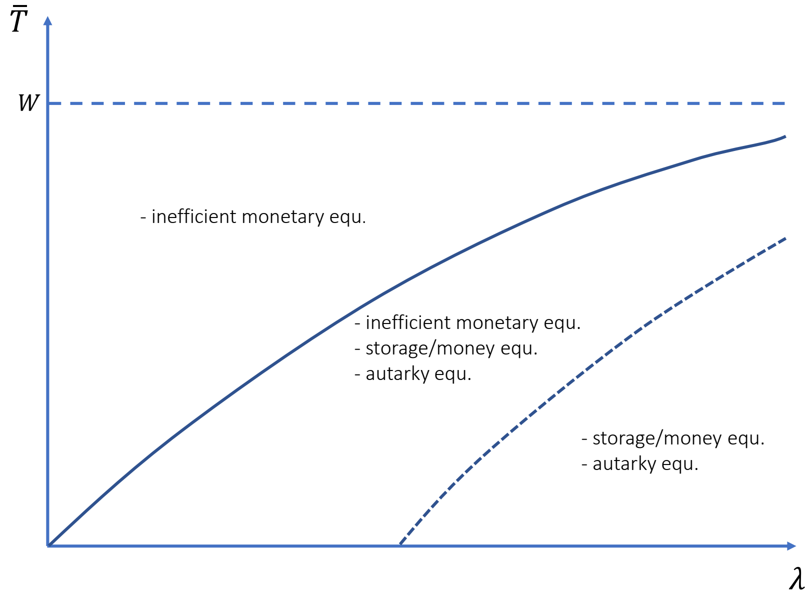

Figure 3 – Equilibria sets in the space $(\lambda, \bar{T})$.

Proof. See Appendix B.5. □

The Proposition shows the role of fiscal capacity in nailing down the set of equilibria. The conditions for the existence of equilibria are illustrated by Figure 3. The set of equilibria crucially depends on the level of $\bar{T}$ – the fiscal capacity of the authority – and λ – the importance of public spending. With fixed taxes, the authority systematically uses its seigniorage power to balance the consumption of the old vis a vis public expenditure. When taxes are fixed too low relative to the importance of public spending, the government can only adjust expenditures to purchase money and, thus, it trades off the welfare gains of money trading with its cost of cutting expenditures. Whenever this equilibrium exists also the possibility of autarky exists as well. The trade-off between public spending and agents' consumption does not arise when taxes can be freely set, as then the government has sufficient tools to adjust government expenditures. In such a case, the authority sustains the value of money to improve the total amount of consumption goods available at that time and set taxes to ensure the fraction that it needs.

It is instructive to remark that, in the extreme case where the government cannot tax and puts no weight on its spending ($\bar{T} = \lambda = 0$), the set of equilibria in Proposition 8 coincides with the set of equilibria in the absence of policy intervention. This means that in the absence of a fiscal counterpart, the authority cannot do better than the market, in line with Wallace (1981b).

Figure 4 illustrates the different types of equilibria. We use the same parameters values of Figure 4 but with fixed taxes at $\bar{T} = 0.056$. This level of taxes is compatible with the existence of an inefficient monetary equilibrium and a money-storage equilibrium.

The inefficient monetary equilibrium is denoted by a solid line with circle markers in figure 4. In this equilibrium money is the only saving asset, but the level of inflation is generically inefficient, increasing in λ and decreasing in $\bar{T}$. Note that, in this equilibrium, the primary fiscal

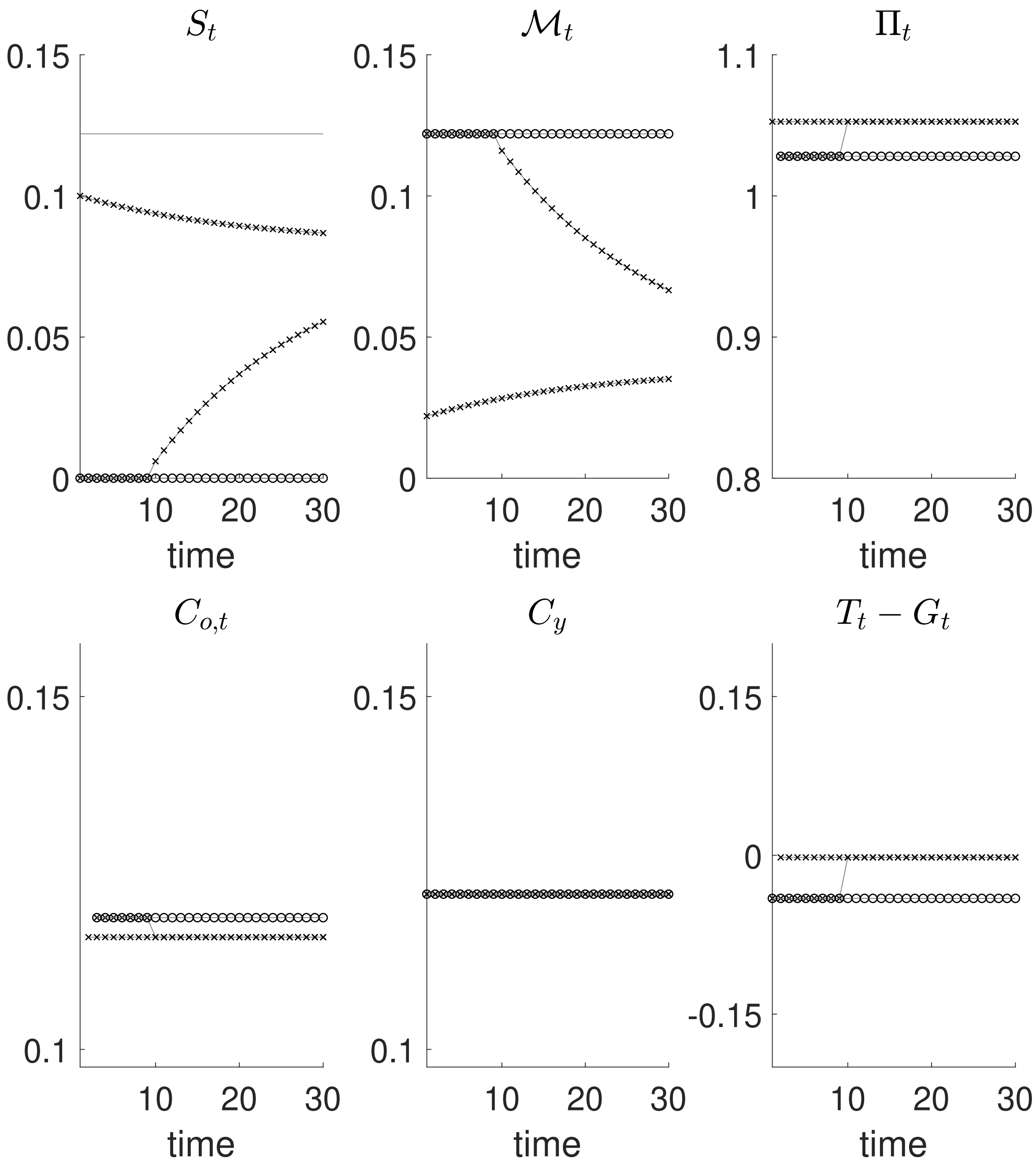

Figure 4 – Equilibria with optimal policy for $\beta = 1$, $\theta = 0.95$, $W = 0.3$, $\lambda = 0.5$ and $\bar{T} = 0.056$. Circles denote the monetary equilibrium, cross markers denote two money-storage equilibria: one that starts at $S_0 = 0.1$ the other at $S_0 = 0$ with a jump at $S_{10} = 0.006$. Autarky, that is not possible in this case, is denoted by a simple dashed line.

surplus is negative, indicating that the authority covers part of its spending by creating and selling money, i.e. generating seigniorage.

Money-storage equilibria are denoted by a solid line with cross markers in figure 4. In these equilibria money and storage are jointly used but money never fully loses value. This is possible because, by selling money, the authority makes inflation equal to θ despite young maintains their real money demand constant. These equilibria obtain for any positive level of storage, no matter if initial or occurring as a jump from the inefficient monetary equilibrium. Both cases are illustrated in the figure, in analogy to figures 1 and 2. In particular, the asymptotic convergence point of any equilibrium where storage is used is the steady state of the money-storage equilibrium in this case.

A Laffer curve of seigniorage. In the storage-money equilibrium, inflation is higher than in the inefficient monetary equilibrium, however, the primary fiscal surplus is *less* negative, showing that actual seigniorage revenues are lower. Effectively, in the storage-money equilibrium, the consumption by both the old and the authority are lower. This means that storage-money equilibrium is the result of a coordination failure between private agents and the authority entailing a sort of Laffer curve of seigniorage.

Let us expand on the reasons behind the coordination failure. Suppose the young decides to save in storage. This action increases the resources available in the next period, and in particular the consumption of the old. Since the authority wants to equalize the marginal utility of its own consumption and the one by the old, it sells money to drain resources from the old. This increases inflation until it matches the return on storage making the young indifferent between saving in money or storage, as in the absence of policy. However, whereas in the absence of policy that equilibrium inflation rate is achieved by subsequent decreases in private money demand, in this equilibrium it is achieved by subsequent increases in money supply. This is what allows real money demand to stay constant at a level lower than in the inefficient monetary equilibrium. In analogy with the Laffer curve of taxation, we can interpret the lower level of real money holdings as a lower "tax base" of seignorage, which pushes the authority to tax more money holdings to extract resources. This higher "tax rate" corresponds to a higher inflation rate. The expectation of such higher inflation rate makes the more intense use of storage self-fulfilling and willingness to tax more through more is self-defeating. The agents would all benefit to be in the monetary equilibrium but, individually, the optimality of their portfolio may lead them to use storage. The authority would also benefit to be in the monetary equilibrium to expand its tax base and obtain a higher revenue from seignorage, but it cannot because of its time-inconsistency.

This inability to resist taxation through seigniorage is also at the base of the existence of the autarky equilibrium. When taxes are too low and, thus, government expenditures are low as well, the government may even have the incentive to drive the price level to negative values so as to tax money holdings. Since negative price levels are not feasible, prices are sent to infinity: such an incentive prevents any credible deflation, which is what the optimal unconstrained policy was able to generate in absence of private demand of money. As a consequence, autarky can be an equilibrium outcome.

The role of private portfolio choices and time-consistency. The presence of the storage technology, and so of a portfolio choice by agents, together with the time-consistent nature of the authority is key for the result on multiplicity; on the contrary the time-consistency of the authority is not sufficient alone. This can be observed in the absence of the storage option: in our model, this is equivalent to setting $\theta = 0$. In this case, Proposition 8 implies that there is only one equilibrium, the monetary inefficient one. The presence of storage choices instead affects the availability of resources, which the authority takes as a given as it acts after the private sector moves. Our finding is then consistent with Nicolini (1996) who shows that a time consistent authority can always implement a single equilibrium, even when paying negligible attention to households' welfare, but only in the absence of a storage technology: this case would correspond to a situation when $\theta = 0$, which, as we underlined, leads also to one equilibrium in our setting.

5.2 Upper bound on taxation

In this subsection, we generalize the findings above by considering a bound on taxation, that is, we explore the case where taxes are fully flexible conditional on being smaller than a certain cap. We show that, when this constraint is sufficiently tight, multiple equilibria can emerge as characterised in Proposition 8, otherwise equilibria as described in Proposition 6 hold.

We now assume that taxes on the young generation have to satisfy:

$$T_t \leq \hat{T}, \tag{35}$$

at any t, with $\hat{T} \geq 0$. Given our result that the optimal time consistent policy is $\mathcal{P}_t = \text{argmax}\{\mathbb{U}_t\}$ at any t in the unconstrained case and $\hat{\mathcal{P}}_t = \text{argmax}\{\mathbb{U}_t\}$ at any t in the constrained case, we obtain that at time t the authority implements $\hat{\mathcal{P}}_t$ when (35) binds and $\mathcal{P}_t$ if not. Leveraging on the results that we have already derived, we can show the following proposition:

Proposition 9. *When*

$$\hat{T} \geq \frac{\lambda + 1 - \beta}{\lambda + 2} W,$$

(i) *the constraint* (35) *does not bind in equilibrium,*

(ii) *a unique equilibrium exists where only money is used as described by Proposition 6.*

Otherwise, when

$$\hat{T} \leq \frac{\lambda + 1 - \beta}{\lambda + 2} W,$$

(i) *the constraint* (35) *always binds in equilibrium,*

(ii) *the set of equilibrium is described by Proposition 8 with taxes T_t fixed at $\hat{T}$;*

in particular, only money is used in equilibrium when also

$$\hat{T} \geq \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta}.$$

Proof. See Appendix B.6 ∎

When its fiscal capacity is constrained, the authority faces a trade-off between monetary stability and its expenditures as in the case of fixed taxes (Proposition 8). When the constraint is tight enough, this trade-off results in multiple equilibria and monetary stability cannot be necessarily ensured.

Indeed, the lack of resources leads the authority to act in conflict with monetary stability: the long run value of monetary stability is not aligned with the short run value of expenditures. This misalignment would even make valuable to commit to keep fixed government expenditures – a commitment that was not necessary with unconstrained taxes.

When this bound is sufficiently large, $\hat{T}/W \geq \lambda/(\lambda+2)$, the monetary equilibrium without money creation is the single equilibrium as in Proposition 6. In this case, the constraint $T_t/W \leq \hat{T}$ does not bind in equilibrium. Interestingly, the constraint can bind off-equilibrium, when $(1+\lambda-\theta)/(2+\lambda) \geq \hat{T}/W \geq \lambda/(\lambda+2)$, but without preventing the interventions to be sufficient to rule out other equilibria, where storage is used.

When the bound becomes tighter, $\hat{T}/W \leq \lambda/(\lambda+2)$, the constraint $T_t \leq \hat{T}$ always binds, in- and off-equilibrium. As a result, we are back to a situation as with fixed taxes, described in Proposition 8. In such a situation, when $\hat{T}/W \geq \lambda\theta/(2+\lambda\theta)$, the monetary equilibrium is still the unique equilibrium, but one in which the authority is creating money so as to finance its expenditures given that, with taxes only, the level of spending would be suboptimal. It is only when $\hat{T}/W < \lambda\theta/(2+\lambda\theta)$ that multiple equilibria may emerge.

6 Conclusion

The question that we have asked in this paper is: to which extent may the pursuit of short-term fiscal objectives sustain long-term monetary stability? The answer provided by this paper is that short-term fiscal objectives can sustain money stability as a by product in so far governments do not have any constraint in taxation, no matter how small is the importance of private agents' utility in the authority objective.

We have extended the basic framework of Sims (2013)'s overlapping generation model by adding a fiscal authority with an explicit objective function. We have derived its optimal time-consistent use of taxation and money operations. We showed that the optimal policy pursuing short-term consumption equality implies that the authority buys money using tax revenues when private agents deviate from an equilibrium where money is the only saving asset. As a result, there exists a single equilibrium.

We have emphasized that the reasons why we obtain a unique equilibrium differ from the ones for which a commitment to positive surpluses can do as suggested by Sims (2013), consistently with the fiscal theory of the price level. Commitments to positive surpluses achieve a unique equilibrium because an equilibrium simply cannot form after a deviation by the private sector. In contrast, our time-consistent optimal policy, let an equilibrium always form after any deviation by the private sector, but, at the same time, it makes such deviation suboptimal.

We then considered limits on the authority's ability to tax and we have showed that these limits may introduce a trade-off between money stability and the financing of the authority's expenditures. When the ability to tax is sufficiently limited, the authority forgoes on maintaining the value of money, thus allowing the possibility of multiple equilibria.

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A Fluctuations in endowment

In this section we will look at the case of time-varying endowment. We will initially look at the dynamics in the absence of policy and then study the optimal policy reaction.

A.1 Absence of policy

Before exploring how time-varying endowment affect the optimal policy, let us review briefly what changes absent policy, i.e. with $\mathcal{P}_t = (0,0,0)$ at each t. In the case of fluctuations in endowment, we have

$$\Pi_{t+1} = \frac{W_t - 2S_t}{W_{t+1} - 2S_{t+1}}.$$

This modification has important consequences on the set of equilibria. As a first result note that the pure monetary equilibrium may not exists any longer. In fact, $S_t = 0$ for any $t \geq 1$ is not an equilibrium when $\Pi_{\tau+1} = W_\tau / W_{\tau+1} \geq \theta^{-1}$. In this case, there not exist an equilibrium where $S_\tau = 0$ because the return on storage *necessarily* exceeds the one on money; on the contrary, S_τ must be strictly positive.

On the other hand, it is possible now an equilibrium where $S_\tau > 0$ and $S_{\tau+1} = 0$. To see this notice that, after substituting $\mathcal{M}_\tau = W_\tau/2 - S_\tau$,(19) now becomes

$$S_{\tau+1} = \theta S_\tau + \frac{W_{\tau+1} - \theta W_\tau}{2},$$

so that, when $\theta W_\tau > W_{\tau+1}$, there exists a value of S_τ, namely

$$0 < \hat{S}_\tau \equiv \frac{\theta W_\tau - W_{\tau+1}}{2\theta} < \frac{W_\tau}{2},$$

such that $S_{\tau+1} = 0$, provided $\hat{S}_\tau < W_\tau/2$. The reason is that, when all savings are in money and the endowment decreases sufficiently fast, the return on money may fall below the return on storage, making storing more attractive instead. In general however, while a jump to positive storage is made necessary in situations of a strong decrease in endowment, the return to pure monetary savings is not. In analogy to the case with constant endowment, a multiplicity of equilibria exists that satisfy where storage is positive also at date $t + 1$.

A.2 Optimal policy reaction with fluctuations in endowment

We build the optimal policy in this case based on two elements.

First, whenever $S_t > 0$ we have that

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1} - (3+\lambda)S_t}{W_{t+1} - \theta(1+\lambda)S_t - S_{t+1}} = \theta^{-1},$$

or

$$S_t = \frac{S_{t+1} + \theta^2 S_{t-1} - W_{t+1} + \theta W_t}{2\theta}, \tag{36}$$

that is, the return on money must be the same than the return on storage.

Second, whenever $S_t = 0$ instead optimal saving choices require

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1}}{W_{t+1} - S_{t+1}} < \theta^{-1},$$

or

$$\theta^2 S_{t-1} < W_{t+1} - \theta W_t - S_{t+1}, \tag{37}$$

that is, the return on money is higher than the return on storage.

These two elements, which do not depend on λ, nail down the unique equilibrium consistent with an optimal policy response. The following proposition characterise the equilibrium path of storage, for arbitrary sequence of endowments, that, for an arbitrary initial S_t at some t, converges to zero storage in a finite time.

Proposition A.1. *For any $\{\lambda, \beta\}$, and a given $S_t \in (0, W_t)$ and sequence of endowments $\{W_\tau\}_{\tau=t+1}^{\infty}$, the sequence of $\{S_\tau\}_{\tau=t+1}^{\infty}$ characterising an equilibrium with optimal policy is such that:*

i) given the unique $n^* \in \mathbb{N}$ that satisfies the following inequality:

$$\frac{n^* W_{t+1+n^*} - \sum_{i=0}^{n^*-1} \theta^{n^*-i} W_{t+1+i}}{\theta^{n^*}} < \theta S_t < \frac{(n^*+1) W_{t+2+n^*} - \sum_{i=0}^{n^*} \theta^{n^*+1-i} W_{t+1+i}}{\theta^{n^*+1}},$$

ii) then there are at least n^* successive storage values $\{S_{t+1+n^*-n}\}_{n=1}^{n^*}$ given by:

$$S_{t+1+n^*-n} = \frac{n\theta^n \theta S_{t+n^*-n} + n\theta^n W_{t+1+n^*-n} - \sum_{i=1}^{n} \theta^{n-i} W_{t+1+n^*-n+i}}{(1+n)\,\theta^n},$$

before the pure monetary state $(S_{t+n^*+1}, S_{t+n^*+2}) = (0, 0)$.

Proof. To build our solution let us suppose that there is a t such that $S_t > 0$, $S_{t+1} = 0$ and $S_{t+2} = 0$ and work backwards. According to (36) we get

$$S_t = \frac{\theta^2 S_{t-1} + \theta W_t - W_{t+1}}{2\theta} \tag{38}$$

from which it is obvious that, to get $S_t > 0$ either W_t is sufficiently big or it must be $S_{t-1} > 0$. Moreover, according to (37) it should be

$$S_t = \frac{\theta^2 S_{t-1} + \theta W_t - W_{t+1}}{2\theta} < \frac{W_{t+2} - \theta W_{t+1}}{\theta^2}$$

i.e.

$$\frac{W_{t+1} - \theta W_t}{\theta} < \theta S_{t-1} < \frac{2W_{t+2} - \theta W_{t+1} - \theta^2 W_t}{\theta^2}. \tag{39}$$

If the inequality is satisfied with $S_{t-1} = 0$ then only at t storage is positive in an equilibrium with $S_{t+1} = S_{t+2} = 0$. Otherwise, it must be that also $S_{t-1} > 0$.

Consider then, $S_{t-1} > 0$. Applying iteratively (38), this requires that

$$S_{t-1} = \frac{S_t + \theta^2 S_{t-2} - W_t + \theta W_{t-1}}{2\theta} = \frac{2\theta^3 S_{t-2} + 2\theta^2 W_{t-1} - \theta W_t - W_{t+1}}{3\theta^2} \tag{40}$$

from which it is obvious that either W_{t-1} is sufficiently big or it must be $S_{t-2} > 0$. Because of (39) and (40) it must be that

$$\frac{2W_{t+1} - \theta W_t - \theta^2 W_{t-1}}{\theta^2} < \theta S_{t-2} < \frac{3W_{t+2} - \theta W_{t+1} - \theta^2 W_t - \theta^3 W_{t-1}}{\theta^3}$$

If the inequality is satisfied with $S_{t-2} = 0$ then from $t-1$ to t storage is positive in an equilibrium with $S_{t+1} = S_{t+2} = 0$. Otherwise, it must be that also $S_{t-2} > 0$.

By iterating we have

$$S_{t+1-n} = \frac{n\theta^n \theta S_{t-n} + n\theta^n W_{t+1-n} - \sum_{i=1}^{n} \theta^{n-i} W_{t+1-n+i}}{(1+n)\,\theta^n} \geq 0,$$

which requires that either W_{t+1-n} is sufficiently large or S_{t-n} must be positive. In particular, it should be

$$\frac{nW_{t+1} - \sum_{i=0}^{n-1} \theta^{n-i} W_{t+1-n+i}}{\theta^n} < \theta S_{t-n} < \frac{(n+1)W_{t+2} - \sum_{i=0}^{n} \theta^{n+1-i} W_{t+1-n+i}}{\theta^{n+1}} \tag{41}$$

If the inequality is satisfied with $S_{t-n} = 0$ then from $t - n + 1$ to t storage is positive in an equilibrium with $S_{t+1} = S_{t+2} = 0$. Otherwise, it must be that also $S_{t-n} > 0$.

The last step of the proof is to verify that

$$\theta S_{t+1-n} = \theta \frac{n\theta^n \theta S_{t-n} + n\theta^n W_{t+1-n} - \sum_{i=1}^{n} \theta^{n-i} W_{t+1-n+i}}{(1+n)\,\theta^n} < \frac{nW_{t+2} - \sum_{i=0}^{n-1} \theta^{n-i} W_{t+2-n+i}}{\theta^{n+1}},$$

leads to (41). This follows from

$$\begin{aligned}
\theta \frac{n\theta^n \theta S_{t-n}}{(1+n)\,\theta^n} &< \frac{nW_{t+2} - \sum_{i=0}^{n-1} \theta^{n-i} W_{t+2-n+i}}{\theta^n} - \theta \frac{n\theta^n W_{t+1-n} - \sum_{i=1}^{n} \theta^{n-i} W_{t+1-n+i}}{(1+n)\,\theta^n} \\
\frac{n\theta^{n+1} \theta S_{t-n}}{(1+n)\,\theta^n} &< \frac{(1+n)nW_{t+2} - (1+n)\sum_{i=1}^{n} \theta^{n+1-i} W_{t+1-n+i} - n\theta^{n+1} W_{t+1-n} + \sum_{i=1}^{n} \theta^{n+1-i} W_{t+1-n+i}}{(1+n)\,\theta^n} \\
\theta S_{t-n} &< \frac{(1+n)nW_{t+2} - n\sum_{i=1}^{n} \theta^{n+1-i} W_{t+1-n+i} - n\theta^{n+1} W_{t+1-n}}{n\theta^{n+1}} \\
\theta S_{t-n} &< \frac{(1+n)W_{t+2} - \sum_{i=0}^{n} \theta^{n+1-i} W_{t+1-n+i}}{\theta^{n+1}}
\end{aligned}$$

which is the same as (41). So we conclude that our recursive formulation indeed holds at any t for given a n.

Now to recover the expressions in the proof we need to operate an appropriate change of variables to express as a given the initial positive level of storage. In practice, whereas the proof defines a sequence of storage expressed as $\{S_{t-n}, S_{t-n+1}, ...S_{t-1}, S_t, 0, 0\}$ the proposition defines the same sequence relabelling time indexes to be $\{S_t, S_{t+1}, ...S_{t+n^*-1}, S_{t+n^*}, 0, 0\}$. □

We finally state here the solution for a sequence of constant endowments, which are the ones considered in the main text. The uniqueness of such paths is established in B.3 (point d.3).

Corollary A.2. *For any $\{\lambda, \beta\}$, and a given $S_t \in (0, W_t)$ and sequence of endowments $W_t = W$ at any t, the sequence of $\{S_\tau\}_{\tau=t+1}^{\infty}$ characterising an equilibrium with optimal policy is such* *that:*

i) given the unique $n^* \in \mathbb{N}$ that satisfies the following inequality:

$$\frac{n^* - \sum_{i=0}^{n^*-1} \theta^{n^*-i}}{\theta^{n^*}} W < \theta S_t < \frac{(n^*+1) - \sum_{i=0}^{n^*} \theta^{n^*+1-i}}{\theta^{n^*+1}} W,$$

ii) then there are at least n^* successive storage values $\{S_{t+1+n^*-n}\}_{n=1}^{n^*}$ given by:

$$S_{t+1+n^*-n} = \frac{n\theta^n \theta S_{t+n^*-n} + \left(n\theta^n - \sum_{i=1}^{n} \theta^{n-i}\right) W}{(1+n)\,\theta^n},$$

before the pure monetary state $(S_{t+n^*+1}, S_{t+n^*+2}) = (0,0)$.

B Proofs

B.1 Proof of Proposition 4

Optimal time-consistent policy. It is simple to find out the rate of inflation Π_t and public consumption G_t that maximize (12). T_t^* is the one that satisfy the budget of the authority.

To see how these are also solutions to the dynamic problem it is sufficient to rewrite the dynamic problem in a recursive formulation as:

$$V(S_{t-1}, S_t) = \max_{\mathcal{P}_t} \left\{ u\left(W_t - G_t - \Pi_t^{-1}\mathcal{M}_{t-1} - S_t\right) + u\left(\Pi_t^{-1}\mathcal{M}_{t-1} + \theta S_{t-1}\right) + \tilde{\lambda} v(G_t) + \beta V(S_t, S_{t+1}) \right\}$$

and notice that $\partial V(S_t, S_{t+1})/\partial \Pi_t = \partial V(S_t, S_{t+1})/\partial G_t = 0$ because S_t is predetermined to the policy choice at time t and S_{t+1} does not respond to any lagged variable.

First best. Let us now show that $\left(\sigma^t_{\mathcal{P}^*}, \sigma^*\right)$ leads to the authority's first best allocation. Let us consider the problem of an unconstrained social planner with the same preferences of the authority. The optimal allocation solves the following problem: for a given S_0 and M_0,

$$\max_{\{C_{o,t}, C_{y,t}, G_t, S_t, M_t, P_t\}_{t\geq 1}} \sum_{t=1}^{\infty} \beta^{t-1} \left(u(C_{o,t}) + u(C_{y,t}) + \tilde{\lambda} u(G_t) \right)$$

$$\begin{aligned}
\text{s.t. } & C_{o,t} - \theta S_{t-1} - \frac{M_{t-1}}{P_t} \leq 0 && (\zeta_t)\\
& C_{y,t} - W_t + S_t + \frac{M_t}{P_t} + T_t \leq 0 && (\mu_t)\\
& \frac{M_{t-1}}{P_t} - \frac{M_t}{P_t} + G_t - T_t \leq 0 && (\gamma_t)\\
& M_t \geq 0 && (\eta_t)\\
& S_t \geq 0 && (\omega_t)\\
& P_t \geq 0 && (\epsilon_t)
\end{aligned}$$

In this problem, we have already take into account the market clearing condition for money.

The first order conditions of this problem are:

$$\begin{aligned}
C_{o,t}: &\qquad \beta^{t-1}u'(C_{o,t}) = \zeta_t \\
C_{y,t}: &\qquad \beta^{t-1}u'(C_{y,t}) = \mu_t \\
G_t: &\qquad \beta^{t-1}\tilde{\lambda}u'(G_t) = \gamma_t \\
S_t: &\qquad \theta\zeta_{t+1} = \mu_t + \omega_{t+1} \\
M_t: &\qquad \frac{-\zeta_{t+1}+\gamma_{t+1}}{P_{t+1}} = \frac{-\mu_t+\gamma_t}{P_t} + \eta_t \\
T_t: &\qquad \mu_t - \gamma_t = 0 \\
P_t: &\qquad (-\zeta_t+\gamma_t)M_{t-1} + (\mu_t-\gamma_t)M_t + \epsilon_t = 0
\end{aligned}$$

where ζ_t, μ_t, γ_t, ω_t, η_t, ϵ_t are Lagrangian associated to the constraints as showed above.

It is inefficient to use only the storage technology, so that $M_t > 0$. In this case, $\eta_t = 0$ and, combining the focs for T_t and M_t, we obtain that at each t:

$$\begin{aligned}
C_{y,t} &= C_{o,t} \\
G_t &= \lambda C_{o,t}
\end{aligned}$$

with $\lambda = 1/(u')^{-1}(\tilde{\lambda})$, are optimal as provided by $\mathcal{P}_t^*$. T_t is then the one satisfying the budget constraint. A social planner would equalize consumption of the young and the old generations and choose public consumption as a fraction of them. Note that a solution also exists in which the consumption of the old and the young are not equalized, money is used and $P_t \to \infty$. Those are the same conditions characterizing the authority's optimal time-consistent policy.

Finally, note that portfolio decisions that are solutions to Lemma 2 are also solutions of the first order conditions. To see this, let us note that

$$\frac{u'(C_{y,t})}{\beta u'(C_{o,t+1})} = \frac{\mu_t}{\zeta_{t+1}} = \rho_{t+1}$$

where ρ_{t+1} is the equilibrium return on savings as defined in the text. This is the same optimality conditions for private saving choices (i.e. on D_t). Let us now turn to portfolio composition. According to the first order condition for S_t, we note that $S_t > 0$, i.e. $\omega_{t+1} = 0$, if and only if $\rho_{t+1} = \theta$. As private agents do, the social planner would use storage only when the return on savings is θ. This demonstrates that $(\sigma^t_{\mathcal{P}^*}, \sigma^*)$ entails the authority's first best allocation.

B.2 Proof of Proposition 5

It is easy to note that $\Pi_{t+1} = 1 < \theta^{-1}$ and $S_t = 0$ for any t is an equilibrium; one in which money is always used and storage never. We refer to this equilibrium as the *pure monetary equilibrium*.

To check if there exist an equilibrium where storage is used jointly with money we should use the arbitrage condition in (2). For $S_t > 0$ at time t we must have $\Pi_{t+1} = \theta^{-1}$. In this case,

(17) obtains as

$$S_{t+1} = \theta S_t + (1-\theta)\frac{W}{2}, \tag{42}$$

which implies $S_{t+1} \geq S_t$, given the limit $S_t \leq W/2$ for each date t. Therefore we obtain that, if storage is used in one period, it must necessarily be used on a larger extent next period. In fact, an equilibrium for each initial level of storage $S_1 \in [0, W/2)$ (S_0 is not an optimal choice, i.e. (17) is not an equilibrium condition for S_0) exists such that storage is always used jointly with money. It is easy to show that in the long run, storage and the real money balance satisfy:

$$\lim_{t\to\infty} S_t = \frac{W}{2} \text{ and } \lim_{t\to\infty} \frac{M_0}{P_t} = lim_{t\to\infty}\mathcal{M}_t = 0 \tag{43}$$

for any initial level of storage S_1, where the latter obtains as a consequence of the former because of the expression of D_t in 2. There are equilibria in which storage is always used, prices grows at a rate $1/\theta$ and money loses value in time until it eventually become worthless; let us call them the *asymptotic autarky equilibria.*

Importantly, all asymptotic autarky equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic autarky equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s < \theta^{-1}$, that is

$$0 \leq S_s < (1-\theta)\frac{W}{2}.$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. What is peculiar of the environment with constant endowment is that once storage is used it will used for ever; this is because, for a given S_t, (17) implies a certain S_{t+1} which has the property $S_{t+1} \geq S_t$, given the limit $S_t \leq W/2$ for each t.

Finally, there also exists a *pure autarky equilibrium* defined as one in which $S_t = W/2$ and $\mathcal{M}_t = M_0/P_t = 0$ for each t in which money is never used and the price level is infinite and grows at a rate larger than $1/\theta$.

B.3 Proof of Proposition 6

To establish our results, we use equations (20) and (21) to investigate how optimally chosen policies affect equilibrium outcomes. We first show that there exists a pure monetary equilibrium where only money is traded. Then we show that neither an asymptotic autarky nor an autarky equilibrium exists. We use this latter result to investigate the continuation of an equilibrium after a deviation of the private sector (i.e. the private sector starts to use the storage technology) and we show that such a deviation is suboptimal.

Before going to the core of the proof, we first show a very useful result:

Lemma B.1. *Suppose that $\theta < 1$, then $R(\theta) < \theta$.*

Proof. $R(\theta) < \theta$ is equivalent to $u^{',-1}(\beta\theta) > 1$, which is true. Indeed, u' is decreasing given that u'' is concave. As a result $u^{',-1}$ is decreasing as well. Thus, given that $\beta\theta < 1$, we have $u^{',-1}(\beta\theta) > u^{',-1}(1)$. Finally, $u'(1)u'(1) = u'(1)$. As a result, $u'(1) = 1$ and $u^{',-1}(1) = 1$. This allows to conclude $u^{',-1}(\beta\theta) > 1$. $\square$

a) The pure monetary equilibrium. Let us first show that there exists an equilibrium where only money is traded. More formally, the pure monetary equilibrium where $S_t = 0$ at each t is an equilibrium. This can be easily seen by checking that $S_t = 0$ at any t implies $\Pi_{t+1} = 1 < \theta^{-1}$ at any t, which are mutually consistent.

b) Non existence of asymptotic autarky equilibria. Let us now show that there cannot be an equilibrium where storage is always used after some date $T \geq 0$. Let us proceed by contradiction. Suppose, indeed, that storage is used in any period $t \geq T$. As a result we have, $\theta = \Pi_{t+1}^{-1}$ and, using (20) and (21), we obtain that storage S_t has to satisfy the following second order differential equation:

$$R(\theta)S_{t+1} - (1 + R(\theta))\theta S_t + \theta^2 S_{t-1} = (R(\theta) - \theta)W.$$

Standard results on second order difference equations point out to S_t has then to converge to $S = W/(1-\theta)$, which implies negative consumption . This is not feasible, and so, there is a contradiction.

c) Non existence of a pure autarky equilibrium. Here we prove that an equilibrium in which real money balance are valueless starting at some date t – $\mathcal{M}_\tau = M_\tau/P_\tau = 0$ starting a $t \geq \tau$ – does not exist either.

Indeed, suppose that there exists a date t such that $\mathcal{M}_t = 0$. Optimal policy at date-$t+1$ is, according to Proposition 4, to set date $t+1$ inflation at 0. This then implies that the return on money is infinite and exceeds the return on storage.

Such 0 inflation rate is implemented when $\mathcal{M}_t = 0$ simply by setting $\mathcal{M}_{g,t+1} > 0$ (and thus $M^S_{t+1} = 0$) at date $t+1$, whatever the value of $\mathcal{M}_{t+1}$.

Moreover, this results extends by continuity to any arbitrarily small deviation which is also profitable: suppose that one agent deviates so that $\mathcal{M}_t = \epsilon$ with $\epsilon > 0$ arbitrarily small. Date-$t+1$ inflation rate is $\Pi_{t+1} = \epsilon/(W - (1+\lambda)\theta S_{t-1} - S_t)$ that is also arbitrarily close to 0. As a result, the return on money also exceeds the return on storage θ, thus making the deviation profitable.

d) No other equilibrium (continuation for given $S_t > 0$). Suppose that $S_t > 0$ is part of an equilibrium. Let us show that this contradicts date-t agent's optimality condition.

For showing this, we first show that the equilibrium continuation after a deviation S_t leads to a path for storage $\{S_t, ..., S_{t+n}\}$ so that S_τ with $\tau \in \{1, .., n-1\}$ is decreasing and $S_{t+n} = 0$, taking S_t as given.

We then show that such a decreasing path leads money to have a strictly better return than storage at date t. That is $S_t > 0$ cannot be part of an equilibrium with constant endowment.

d.1.) The equilibrium continuation. Let us show that there exists a continuation equilibrium. For this, it is sufficient to show that there exists a path for storage fo a given

$S_t > 0$ and satisfying equilibrium conditions. In appendix A we give a closed form solution to such a path, generalizing to time-varying endowment.

First, suppose that such an equilibrium path for storage exists. Then, there exists $n' > 0$ such that $S_{t+n'} = 0$. Suppose that it is not the case, then S_τ converges to $S = W/(1-\theta)$, which is not feasible. Let us denote by n the smallest value such that $S_{t+n} = 0$.

Second, let us show that there exists a unique path $\{S_t, ..., S_{t+n}\}$ such that $S_{t+n} = 0$. First, let us suppose that n exists. At any date $\tau \in \{t+1, ...t+n-1\}$,

$$R(\theta)S_{\tau+1} - (1 + R(\theta))\theta S_\tau + \theta^2 S_{\tau-1} = R(\theta)W - \theta W,$$

This defines a linear difference equation of order 2 for S_τ and $S_{t+n} = 0$ as well as S_t define two boundary conditions. As a result, there exists a unique path $\{S_{t+1}, ..., S_{t+n-1}\}$ solving the linear difference equation combined with the boundary conditions.

We can show that there also exists a unique n such that $\{S_t, ..., S_{t+n}\}$ is an equilibrium path – which then implies that there exists a unique equilibrium path $\{S_t, ..., S_{t+n}\}$. However, we do not need this argument for this proof and we relegate the proof of uniqueness at the end of this proof.

Our objective here is showing that any potential continuation of equilibrium leads to a decreasing path for storage. More precisely, let us show that the sequence of S_τ is decreasing:

Lemma B.2. *Suppose that there exists $t + n$ such that $S_{t+n} = 0$ and for all τ such that $t < \tau < t + n$*

$$R(\theta)S_{\tau+1} + \theta^2 S_{\tau-1} - (R(\theta) - \theta)W = \theta(1 + R(\theta))S_\tau$$

then $S_t > S_{t-1} > ... > S_{t+n} = 0$. In addition, we have that $S_\tau < \theta S_{\tau-1}$ for all $\tau \in \{t+1, t+n\}$.

Proof. We proceed by iteration. Let us first show that $S_{t+n-2} > S_{t+n-1}$. At date $t + n - 1$, we have

$$R(\theta)S_{t+n} + \theta^2 S_{t+n-2} - (R(\theta) - \theta)W = \theta(1 + R(\theta))S_{t+n-1}$$

Using the fact that $S_{t+n} = 0$, we can then write:

$$\frac{\theta}{1 + R(\theta)} S_{t+n-2} - \frac{R(\theta) - \theta}{\theta(1 + R(\theta))} W = S_{t+n-1}$$

Given

$$\frac{\theta}{1 + R(\theta)} < 1 \quad \text{and} \quad \frac{R(\theta) - \theta}{\theta(1 + R(\theta))} W > 0,$$

then $S_{t+n-1} < S_{t+n-2}$.

Suppose that that $S_{t+n-1} < S_{t+n-2} < ... < S_\tau$. Let us show that $S_{\tau-1} > S_\tau$. We can write at date τ:

$$\frac{R(\theta)}{\theta} S_{\tau+1} + \theta S_{\tau-1} - \frac{R(\theta) - \theta}{\theta} W = (1 + R(\theta))S_\tau$$

Given that $S_{\tau+1} \leq S_t$, we obtain:

$$\frac{\theta}{R(\theta)} S_{\tau-1} - \frac{R(\theta) - \theta}{R(\theta)\theta} W \geq S_\tau$$

which implies that $S_\tau < \theta S_{\tau-1}$. $\square$

d.2.) Optimal portfolio decision at date t. Let us now investigate what is the implication of having a decreasing path for storage after a deviation S_t.

If $\theta^2 S_t \leq (R(1) - \theta)W$, then at date $t + 1$, storage is not used. At date t, the return on money is then:

$$\frac{P_t}{P_{t+1}} = R_t(\rho_{t+1}) \frac{W - (1+\lambda)\theta S_t}{W - (1 + R_t(\rho_{t+1})(2+\lambda))S_t}.$$

Let us show that this return strictly exceeds θ.

$$R_t(\rho_{t+1}) \frac{W - (1+\lambda)\theta S_t}{W - (1 + R_t(\rho_{t+1})(2+\lambda))S_t}.$$

Suppose now that $\theta^2 S_t > (R(1) - \theta)W$. At date t, the return on money is then:

$$R_t(\theta) \frac{W - (1+\lambda)\theta S_t - S_{t+1}}{W - (1 + R_t(\theta)(2+\lambda))S_t}$$

Using Lemma B.2, $S_{t+1} < \theta S_t$ and thus:

$$W - (1+\lambda)\theta S_t - S_{t+1} \geq W - ((1+\lambda)\theta + \theta)S_t \geq W - (1 + R_t(\theta)(2+\lambda))S_t$$

as $R_t(\theta) > 0$. This implies that

$$R_t(\theta) \frac{W - (1+\lambda)\theta S_t - S_{t+1}}{W - (1 + R_t(\theta)(2+\lambda))S_t} > R_t(\theta).$$

Finally, as Lemma B.1 implies that $R_t(\theta) > \theta$, we can conclude that using storage is suboptimal at date t.

d.3.) Uniqueness of the equilibrium continuation after S_t. Let us show that there exists a unique continuation of an equilibrium after S_t.

For every integer n', we can obtain a unique sequence $\{S_t, ..., S_{t+n'}\}$. However, in an equilibrium, the sequence should also be such that $S_{t+n'} = 0$ is optimal, which requires $S_{t+n'-1}$ to satisfy $\theta^2 S_{t+n'-1} < (R(1) - \theta)W$.

Let us show that there exists a unique n, such that $\theta^2 S^n_{t+n-1} < (R(1) - \theta)W$. To this purpose, let us show the following lemma:

Lemma B.3. *S^n_{t+n-1} is decreasing with n.*

Proof. First, let us show that $S^1_t = S_t > S^2_{t+1}$. Indeed, $\theta^2 S_t - (R(\theta) - \theta)W = \theta(1 + R(\theta))S^2_{t+1} > S_t$.

Let us extend this proof to n. To this purpose, let us note that:

$$\theta^2 S^n_{t+n-2} - (R(\theta)-\theta)W = \theta(1+R(\theta))S^n_{t+n-1}$$
$$\theta^2 S^{n+1}_{t+n-1} - (R(\theta)-\theta)W = \theta(1+R(\theta))S^{n+1}_{t+n}$$

As a result, $S^{n+1}_{t+n} < S^n_{t+n-1}$ if and only $S^{n+1}_{t+n-1} > S^n_{t+n-2}$. Let us investigate whether $S^{n+1}_{t+n-1} > S^n_{t+n-2}$. To this purpose, let us note that:

$$R(\theta)(S^{n+1}_{t+n} - S^n_{t+n-1}) + \theta^2(S^{n+1}_{t+n-2} - S^n_{t+n-3}) = \theta(1+R(\theta))(S^{n+1}_{t+n-1} - S^n_{t+n-2})$$
$$(1+R(\theta))(S^{n+1}_{t+n} - S^n_{t+n-1}) = \theta(S^{n+1}_{t+n-1} - S^n_{t+n-2})$$

We can infer two results from these equations. On the one hand, there exists $A_{t+n-1}(\theta) > 1$ such that $A(\theta)(S^{n+1}_{t+n-1} - S^n_{t+n-2}) = (S^{n+1}_{t+n-2} - S^n_{t+n-3})$. On the other hand, $S^{n+1}_{t+n-1} > S^n_{t+n-2}$ if and only if $S^n_{t+n-3} > S^{n+1}_{t+n-2}$.

Let us proceed by iteration: suppose that there exists $A_\tau(\theta) > 1/\theta$ such that for some τ:

$$R(\theta)(S^{n+1}_{\tau} - S^n_{\tau-1}) + \theta^2(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = \theta(1+R(\theta))(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$
$$A(\theta)(S^{n+1}_{\tau} - S^n_{\tau-1}) = \theta(S^{n+1}_{\tau-1} - S^n_{t+n-2})$$

We then obtain:

$$\theta^2(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = \theta\left((1+R(\theta)) - \frac{R(\theta)}{A(\theta)}\right)(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$

As a result there exists $A_{\tau-1}(\theta) > 1$ such that:

$$(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = A_{\tau-1}(\theta)(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$

In the end, we obtain by iteration that

$$S^{n+1}_{t+n-1} - S^n_{t+n-2} = A_{t+n-1}(\theta) \times .... \times A_{t+1}(\theta)(S^{n+1}_{t+1} - S_t)$$

Given that all the As are positive and $S_t > S^{n+1}_{t+1}$, we then obtain that $S^{n+1}_{t+n} < S^n_{t+n-1}$. As a result, S^n_{t+n-1} is a decreasing function of n. □

Given that for all τ, we have

$$\frac{\theta}{R(\theta)}S_{\tau-1} - \frac{R(\theta)-\theta}{R(\theta)\theta}W \geq S_\tau,$$

we can find a sufficiently large n' such that $\theta^2 S^{n'}_{t+n'-1} < (R(1)-\theta)W$. Using Lemma B.3, there exists a unique n such that the sequence of storage decisions $\{S_t, ..., S_{t+n}\}$ is such that $\theta^2 S^n_{t+n-1} < (R(1)-\theta)W$ and $\theta^2 S^n_\tau > (R(1)-\theta)W$ at any previous dates.

B.4 Proof of Proposition 7

Let us show that the inflation rate cannot be different from Π in equilibrium.

To start with, if $\Pi_t^{-1} \leq 1$ for any date t, then $\mathcal{M}_t$ is a strictly decreasing sequence that ultimately becomes negative, which cannot be so an equilibrium has to be such that $\Pi_t^{-1} > 1$ at least for some periods.

Suppose that $\Pi_t^{-1} < \bar{\Pi}^{-1}$. As a result, $\mathcal{M}_t < (W-T)/2$ and $S_t > 0$. As a result $\Pi_{t+1}^{-1} = \theta$. In turn, $\mathcal{M}_{t+1} < \mathcal{M}_t$ and $S_{t+1} > 0$ so that $\Pi_{t+2}^{-1} = \theta$. By iterating this reasoning, one can see that $\Pi_t^{-1} > 1$ and $\mathcal{M}_t$ becomes ultimately negative, which cannot be.

Suppose that $\Pi_t^{-1} > \bar{\Pi}^{-1}$. This happens only when $\mathcal{M}_{t-1} = (W-T)/2$ but it also implies that $\mathcal{M}_t > (W-T)/2$, which cannot be. As a result, $\Pi_t = \bar{\Pi}$ is the only solution.

B.5 Proof of Proposition 8

The proof works as follows. First, as in the benchmark case, we find equations that equilibrium variables have to solve. Then we use these equations to provide conditions under which different equilibria may arise.

Equilibrium characterization. Using (30), we get the actual law of motion of inflation of the real value of savings and inflation as:

$$\mathcal{M}_t = \frac{W-\bar{T}}{1+R(\rho_{t+1})} - S_t \tag{44}$$

$$\Pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{(1+\lambda)\left(W-\bar{T}\right) - (1+R(\rho_{t+1}))(1+\lambda)S_t}{W+\bar{T} - (1+R(\rho_{t+1}))\left(\lambda\theta S_t + S_{t+1}\right)} \tag{45}$$

provided $W+\bar{T} \geq (1+R(\rho_{t+1}))(\lambda\theta S_t + S_{t+1})$, otherwise we have $\mathcal{M}_{t+1} \to 0$ and $\Pi_{t+1} \to \infty$. We are ready now to investigate investigate how optimally chosen policies affect equilibrium outcomes.

The pure monetary equilibrium. The pure monetary equilibrium where $S_t = 0$ at each t is still an equilibrium provided $(1+\lambda)(W-\bar{T})/(W+\bar{T}) < \theta^{-1}$. This can be easily seen by checking that $S_t = 0$ at any t implies $\Pi_{t+1} = (1+\lambda)(W-\bar{T})/(W+\bar{T})$ at any t from (45). In turn, $S_t = 0$ requires that $\Pi_{t+1} \leq \theta^{-1}$, thus implying that $(1+\lambda)(W-\bar{T})/(W+\bar{T})$ does not exceed θ^{-1}. We then obtain $\mathcal{M}_t$ from (27) with $S_t = 0$ and G_t from (28).

In case $\Pi_{t+1} > \theta^{-1}$ implies $S_t > 0$, so that a pure monetary equilibrium does not exist in that case.

Existence of asymptotic storage equilibria. We investigate now whether there are equilibria where both money and storage are used. $S_t > 0$ implies $\Pi_t = \theta^{-1}$ at t that, is:

$$S_t = \theta S_{t-1} + \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W-\bar{T})}{1+R(\theta)}$$

Let us first consider the case $\theta(1+\lambda)(W-\bar{T}) < W + R(\theta)\bar{T}$. In such a case, $S_t > 0$ implies $S_{t+\tau} > 0$ for $\tau \geq 1$. However, an equilibrium where $S_t > 0$ for each $t \geq \tau$ requires a sequence $\{S_t\}_{t=1}^{\infty}$ converging monotonically to

$$\bar{S} = \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W-\bar{T})}{(1+R(\theta))(1-\theta)}.$$

As previously noted, to be feasible, $\bar{S}$ should satisfy $\bar{S} \leq \left(W - \bar{T}\right)/(1+R(\theta))$. As a result, a necessary condition to be an equilibrium is:

$$\bar{T} \leq \frac{\lambda\theta}{1+R(\theta)+\lambda\theta}W.$$

Otherwise, an equilibrium where money and storage are jointly used does not exist.

Similarly to the case without any policy, all asymptotic storage equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic storage equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s \leq \theta^{-1}$, that is

$$0 \leq S_s < \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W-\bar{T})}{1+R(\theta)}.$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. Also here once storage is used it will used for ever.

In the case when $\theta(1+\lambda)(W-\bar{T}) > W + R(\theta)\bar{T}$, the sequence of storage S_t converges to a negative value; however this violates the constraint $S_t \geq 0$. Thus, in this case, an equilibrium where storage is used with money does not exist.

Finally, let us note that when condition

$$\bar{T} < \frac{\theta\lambda}{1+R(\theta)+\theta\lambda}W$$

is satisfied, the condition $(1+\lambda)(W-\bar{T})/(W+\bar{T}) \leq \theta^{-1}$ is also satisfied. Indeed, this latter condition is a decreasing function of $\bar{T}$ and the condition is satisfied for $\bar{T} = \frac{\theta\lambda}{2+\theta\lambda}W > \frac{\theta\lambda}{1+R(\theta)+\theta\lambda}W$.

Existence of pure autarky equilibria. We study here the conditions for the existence of a pure autarky equilibrium – i.e. one in which $\mathcal{M}_t = 0$ for any t. Without loss of generality, we consider period 1. Suppose that $\mathcal{M}_1 = 0$. The optimal rate of inflation at date 2 is:

$$\Pi_2 = \frac{(1+\lambda)\mathcal{M}_1}{\frac{W+R(\theta)\bar{T}}{1+R(\theta)} - \lambda\theta S_1 - S_2}$$

$$\Pi_2 = \frac{(1+\lambda)(1+R(\theta))\mathcal{M}_1}{W + R(\theta)\bar{T} - (\lambda\theta+1)(W-\bar{T})}$$

using the fact that, In autarky, $S_1 = S_2 = (W - \bar{T})/(1 + R(\theta))$. The denominator is strictly positive when:

$$\bar{T} > \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W,$$

in which case $\Pi_2 = 0$ when $\mathcal{M}_1 = 0$. As a result, agents are strictly better off not to store. Otherwise, when

$$\bar{T} \leq \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W,$$

B.6 Proof of Proposition 9

To prove this Proposition, we first determine conditions under which the constraint on T_t may bind. We split this investigation depending on whether storage is used in equilibrium.

(Potential) equilibria without storage. Suppose that storage is never used in equilibrium. From Proposition 6, the unconstrained level of taxes is $(1-\beta+\lambda)/(\lambda+2)W$. Depending on the value of $\hat{T}$, this level of taxes is then constrained by $\hat{T}$. When $\hat{T} \geq (1-\beta+\lambda)/(\lambda+2)W$, there exists a monetary equilibrium as described by Proposition 6. Otherwise, a monetary equilibrium exists under the condition of Proposition 8.

(Potential) Equilibria with storage. Suppose that storage is used at some date. Let us first show the following lemma.

Lemma B.4. *At date t, suppose that storage is used and that the constraint binds:*

$$\frac{\lambda + 1 - R(\theta)^{-1}}{2 + \lambda} W + \frac{1 + R(\theta)^{-1}}{2 + \lambda}(S_t - \theta S_{t-1}) \geq \hat{T} \tag{46}$$

Then the constraint binds for $t+1$.

Proof. Suppose that storage is used at date t. As a result $S_t > 0$ and the expected return is θ.

Given that the constraint binds at date t, we can use $R(\theta)(S_{t+1} - \theta S_t) - \theta(S_t - \theta S_{t-1}) = (R(\theta) - \theta)W$ to replace $(S_t - \theta S_{t-1})$ in the constraint to obtain:

$$\frac{(\lambda + 1 - R(\theta)^{-1})}{2 + \lambda} W + \frac{(1 + R(\theta)^{-1})}{2 + \lambda} \left(\frac{R(\theta)}{\theta}(S_{t+1} - \theta S_t) - \left(\frac{R(\theta)}{\theta} - 1 \right) W \right) \geq \hat{T} \tag{47}$$

As a result, the constraint also binds when:

$$\left(\frac{R(\theta)}{\theta} - 1 \right)(S_{t+1} - \theta S_t - W) \tag{48}$$

This is always satisfied as $R(\theta) > \theta$ and $S_t < W/(1-\theta)$. □

As a result of Lemma B.4, either the constraint never binds during or after storage is used or it always binds. If it never binds along these paths, the proof of Proposition 6 implies that these paths cannot be an equilibrium outcome. If the constraint always binds, Proposition 8 implies that these paths can be equilibrium outcomes only when storage is always used and that:

$$\hat{T} \leq \frac{\theta(1+\lambda)-1}{R(\theta)+\theta(1+\lambda)}W. \tag{49}$$

Finally, note that when $\hat{T} \geq \frac{1-\beta+\lambda}{\lambda+2}$, no such paths can be equilibrium outcomes as:

$$\hat{T} \geq \frac{1-\beta+\lambda}{\lambda+2}W \geq \frac{\lambda}{2+\lambda}W \geq \frac{\theta(1+\lambda)-1}{R(\theta)+\theta(1+\lambda)}W, \tag{50}$$

C Micro-foundations for money purchases

In this appendix, we investigate two motives that make money purchases preferred to direct transfers to old households. First, when agents are heterogeneous, but taxes are not contingent to such heterogeneity, money purchases is a more efficient policy instrument of direct transfers; we show that this heterogeneity can be arbitrarily small, still the principle holds. Second, when implementing direct transfers is costly, we show that money purchases are preferred to direct transfers only when the cost of transfers to the old is positive irrespective on whether the cost of transfers to the young is positive.

Preference heterogeneity To begin with, agents can differ in their preferences. This can translate into heterogeneous savings. Let us elaborate an example of such heterogeneity.

Let us assume that agents' preferences are as follows: $u(C^i_{y,t}, C^i_{o,t}) = \log C^i_{y,t} + \gamma_i \log C^i_{o,t}$ with heterogeneous γ_i. We also assume that a group of mass p of agents are such that $\gamma_i = 1$ – *savers*, in which case $i = s$ – and the rest are such that $\gamma_i = 0$ – *consumers*, in which case $i = c$. The former agents save half of their endowment net of taxes to be consumed in the second period of their life – as in the benchmark model – , while the latter do not save at all.

As a result, consumption of savers while being young are:

$$C^s_{y,t} = \mathcal{M}^s_t + S^s_t = \frac{W - T_{y,t}}{2} \text{ and } C^c_{y,t} = W - T_{y,t},$$

where $C^s_{y,t}$ is the consumption of savers and $C^s_{y,t}$ the consumption of consumers. The government's budget constraint is:

$$T_{y,t} + T_{o,t} + \frac{M^S_{g,t}}{P_t} = \mathcal{M}_{g,t}$$

and, thus:

$$C^s_{y,t} = \mathcal{M}^s_t + S^s_t = \frac{W - \mathcal{M}_{g,t} + \frac{M^S_{g,t}}{P_t} + T_{o,t}}{2} \text{ and } C^c_{y,t} = W - \mathcal{M}_{g,t} + \frac{M^S_{g,t}}{P_t} + T_{o,t}.$$

Integrating the first equality across all savers yields:

$$\mathcal{M}_t + S_t = p\frac{W - \mathcal{M}_{g,t} + \frac{M^S_{g,t}}{P_t} + T_{o,t}}{2}.$$

Using the market clearing condition for the money market, we obtain:

$$\mathcal{M}_{g,t} + \mathcal{M}_t = \frac{M^S_{g,t}}{P_t} + \frac{M_{t-1}}{P_t}.$$

and thus:

$$\mathcal{M}_t + S_t = p\frac{W - \frac{M_{t-1}}{P_t} + \mathcal{M}_t + T_{o,t}}{2}.$$

We then obtain that:

$$\mathcal{M}_t = \frac{p}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t}\right) - \frac{2}{2-p}S_t$$

We can plug this value into the expressions for agents' consumption levels so that the current stock of money M_t disappears:

$$C^s_{y,t} = \mathcal{M}_t + S_t = \frac{1}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t\right) \tag{51}$$

$$C^c_{y,t} = \frac{2}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t\right) = 2c^S_{y,t} \tag{52}$$

The resulting problem for the authority is:

$$\max_{\Pi_t, T_{o,t}} \left\{\int \log\left(c^i_{y,t}\right) di + \int \log\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{i,t-1} - T_{o,t}\right) di\right\}.$$

The first order conditions with respect to Π_t and $T_{o,t}$ are as follows:

$$\frac{\mathcal{M}_{t-1}}{W - \mathcal{M}_{t-1} + T_{o,t} - S_t} = \int \frac{\gamma_i \mathcal{M}^i_{t-1}}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} - T_{o,t}} di$$

$$\frac{1}{W - \mathcal{M}_{t-1} + T_{o,t} - S_t} = \int \frac{\gamma_i}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} - T_{o,t}} di$$

with $C^s_{y,t}$ and $C^c_{y,t}$ defined by equations (51) and (52).

Let us compute the right hand sides of the two conditions:

$$\int \frac{\gamma_i \mathcal{M}^i_{t-1}}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} - T_{o,t}} di = \frac{\mathcal{M}_{t-1}}{1/p\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}}$$

$$\int \frac{\gamma_i}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} - T_{o,t}} di = \frac{p}{1/p\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}}$$

The first order conditions can then be written:

$$\frac{1}{W - \mathcal{M}_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \frac{1}{1/p\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}}$$
$$\frac{1}{W - \mathcal{M}_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \frac{p}{1/p\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}}$$

Both conditions cannot hold at the same time as soon as $p < 1$, which implies that only:

$$\frac{1}{W - \mathcal{M}_{t-1}\Pi_t^{-1} - S_t} = \frac{p}{\left(\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right)}$$

may bind in equilibrium. In particular, that means that there exists no interior solution for $T_{o,t}$ that has to equal 0. As a result of these conditions, we obtain the following expression for $\mathcal{M}_{t-1}\Pi_t^{-1}$:

$$\mathcal{M}_{t-1}\Pi_t^{-1} = \frac{pW - \theta S_{t-1} - pS_t}{1+p},$$

which allows to also rewrite $\mathcal{M}_t$ as follows:

$$\mathcal{M}_t = \frac{1}{(2-p)(1+p)}\left(pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t\right).$$

The inflation rate at $t+1$ can be expressed as function of storage. Using the no-arbitrage condition between money and storage, we find:

$$\frac{pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t}{pW - \theta S_t - pS_{t+1}} \frac{1}{2-p} = \theta^{-1}$$

which leads to:

$$(2-p-\theta)W = (2-p)S_{t+1} + \theta(p-3)S_t + \theta^2 S_{t-1}.$$

As in the benchmark case, the sequences S_t satisfying this equation are of the following form, for $p < 1$:

$$S_t = \lambda_1\theta^t + \lambda_2\left(\frac{\theta}{2-p}\right)^t + \frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2}W$$

As θ and $\theta/(2-p)$ are both below 1, S_t converges to $\frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2}W$. Given that $\theta(p-2)+\theta^2 = \theta(\theta+p-2) < 0$, we then obtain that S_t is ultimately above $W/2$. We can then use the same logic as for the proof of Proposition 6.

Costly transfers Let us introduce two costs in our benchmark model. First, the cost of transferring $T_{o,t}$ to old agents is $(1+\nu)T_{o,t}$. Second, we assume that the cost of raising $T_{y,t}$ amount of resources cost $(1+\lambda)T_{y,t}$ to young agents.

We plug the government budget constraint $T_{y,t} + \frac{M_{g,t}^S}{P_t} = \mathcal{M}_{g,t} + (1+\nu)T_{o,t}$ into individual saving decisions:

$$C_{y,t}^i = \mathcal{M}_t^i + S_t^i = \frac{W - (1+\lambda)T_{y,t}}{2}$$

to obtain these individual saving decisions as follows:

$$2\mathcal{M}_t^i + 2S_t^i = W - (1+\lambda)\mathcal{M}_{g,t} + (1+\lambda)\frac{M_{g,t}^S}{P_t} + (1+\lambda)(1+\nu)T_{o,t}$$

Integrated over i, this condition yields:

$$2\mathcal{M}_t + 2S_t = W - (1+\lambda)\mathcal{M}_{g,t} + (1+\lambda)\frac{M_{g,t}^S}{P_t} + (1+\lambda)(1+\nu)T_{o,t}$$

Using the market clear condition for the money market, we obtain:

$$\frac{M_{t-1}}{P_t} - \mathcal{M}_t = \mathcal{M}_{g,t} - \frac{M_{g,t}^S}{P_t}$$

and we can infer that:

$$\mathcal{M}_t = \frac{W - (1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)(1+\nu)T_{o,t} - 2S_t}{1-\lambda}$$

As a result, using the fact that $M_{t-1} = P_{t-1}\mathcal{M}_{t-1}$, we obtain that:

$$C_{y,t}^i = \mathcal{M}_t^i + S_t^i = \frac{W - (1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1} + (1+\lambda)(1+\nu)T_{o,t} - (1+\lambda)S_t}{1-\lambda}$$

Except for this, agents are homogeneous. The problem can be rewritten as:

$$\max_{\Pi_t, T_o} \int \log\left(\frac{W - (1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1} + (1+\lambda)(1+\nu)T_{o,t} - (1+\lambda)S_t}{1-\lambda}\right) \mathrm{di} + \cdots$$
$$\cdots + \int \log\left(\mathcal{M}_{t-1}^i\Pi_t^{-1} + \theta S_{t-1}^i - T_{o,t}\right) \mathrm{di},$$

The first order conditions with respect to Π_t and $T_{o,t}$ are:

$$\frac{\mathcal{M}_{t-1}(1+\lambda)}{W - (1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1} + (1+\lambda)T_{o,t}(1+\nu) - (1+\lambda)S_t} \geq \int \frac{\mathcal{M}_{t-1}^i}{\mathcal{M}_{t-1}^i\Pi_t^{-1} + \theta S_{t-1}^i - T_{o,t}} di$$
$$\frac{(1+\nu)(1+\lambda)}{W - (1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1} + (1+\lambda)T_{o,t}(1+\nu) - (1+\lambda)S_t} \geq \int \frac{1}{\mathcal{M}_{t-1}^i\Pi_t^{-1} + \theta S_{t-1}^i - T_{o,t}} di$$

Let us then show the following:

Proposition C.1. *When $\nu > 0$, it is optimal not to make transfers to the old generation ($T_{o,t} = 0$).*

Proof. Let us show that if the first condition binds, then the second condition is satisfied. If the first condition binds, it can be rewritten as:

$$\frac{(1+\lambda)}{W-(1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1}+(1+\lambda)T_{o,t}(1+\nu)-(1+\lambda)S_t}=\int\frac{\frac{\mathcal{M}_{t-1}^i}{\mathcal{M}_{t-1}}}{\mathcal{M}_{t-1}^i\Pi_t^{-1}+\theta S_{t-1}^i-T_{o,t}}di$$

Given that $\mathcal{M}_{t-1}^i$ is positively correlated with $\mathcal{M}_{t-1}^i\Pi_t^{-1}+\theta S_{t-1}^i-T_{o,t}$ when $\Pi_t^{-1}\geq\theta$, we obtain:

$$\frac{(1+\lambda)}{W-(1+\lambda)\mathcal{M}_{t-1}\Pi_t^{-1}+(1+\lambda)T_{o,t}(1+\nu)-(1+\lambda)S_t}\geq\int\frac{1}{\mathcal{M}_{t-1}^i\Pi_t^{-1}+\theta S_{t-1}^i-T_{o,t}}di$$

As a result, when $\nu\geq 0$, the second condition is satisfied as well. A sufficient condition then for the two constraints not to bind at the same time is then than $\nu>0$, thus implying that $T_{o,t}=0$. □

Note that the cost of raising taxes on the young, λ has a symmetric effect on the two first order conditions, indicating that direct transfers can be ruled out not because of the cost of raising resources but because of the relative cost of transfers over money purchases, as captured by ν. When the cost of raising resources satisfies $\lambda=0$ as in the benchmark model, the optimality condition for money purchases leads to the same solution as Proposition 4.

In the end, money purchases are preferred to direct transfers only when the cost of transfers to the old (ν) is positive but not when only the cost of transfers to the young (λ) is positive. This then implies that the frictions that lead to money purchases have to increase to cost of transferring resources but not the cost of raising resources, which affects both direct transfers and money purchases.

D Randomization of portfolios

In the case where agents are indifferent between storage and money, they may randomize portfolios so that these portfolios are heterogeneous. In this appendix, we show that such a randomization does not affect our results.

First, let us find the consumption level of a young agent i. To this purpose, we combine the government budget constraint with the market clearing condition for money to obtain

$$T_t=\mathcal{M}_{g,t}-\frac{M_{g,t}^S}{P_t}=\frac{M_{t-1}}{P_t}-\mathcal{M}_t$$

into individual saving decisions:

$$C_{y,t}^i=\mathcal{M}_t^i+S_t^i=\frac{W-T_t}{2}\tag{53}$$

to obtain these individual saving decisions as follows:

$$2\mathcal{M}_t^i+2S_t^i=W-\frac{M_{t-1}}{P_t}+\mathcal{M}_t.$$

Integrated over i, this condition yields:

$$\mathcal{M}_t = W - \frac{M_{t-1}}{P_t} - 2S_t$$

and, thus:

$$C^i_{y,t} = \mathcal{M}^i_t + S^i_t = W - \frac{M_{t-1}}{P_t} - S_t$$

To show the logic of our argument, we explore here the simple case with $\lambda = 0$, $u(\cdot) = \log(\cdot)$ and directly take the point fo view of a myopic authority. This leads to the following optimization problem:

$$\max_{\Pi_t} \left\{ \int \log\left(W - \mathcal{M}_{t-1}\Pi_t^{-1} - S_t\right) \mathrm{di} + \int \log\left(\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1}\right) \mathrm{di} \right\},$$

Note that the young generation consumes the same, no matter its portfolio choice, consistently with young agents' indifference between portfolios.

The first order conditions with respect to Π_t is:

$$\frac{\mathcal{M}_{t-1}}{W - \mathcal{M}_{t-1}\Pi_t^{-1} - S_t} = \int \frac{\mathcal{M}^i_{t-1}}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1}} di \tag{54}$$

Interestingly, (54) can be rewritten in a more compact way:

$$cov\left(\mathcal{M}^i_{t-1}, \frac{1}{\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1}}\right) = 0$$

In equilibrium, if agents are indifferent between storage and money, a no-arbitrage condition should hold on asset returns: $\theta = P_{t-1}/P_t$. Using (53), this implies that

$$\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} = \theta\left(\mathcal{M}^i_{t-1}\Pi_t^{-1} + S^i_{t-1}\right) = \theta\frac{W - T_t}{2},$$

which implies that $\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1}$ is constant across individuals. Integrating this condition over households and using the fact that there is a mass 1 of them, we obtain that:

$$\mathcal{M}^i_{t-1}\Pi_t^{-1} + \theta S^i_{t-1} = \mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}.$$

As a result, in equilibrium, equation (54) simplifies so that we obtain the same first order condition as in the homogeneous case:

$$\frac{\mathcal{M}_{t-1}}{W - \mathcal{M}_{t-1}\Pi_t^{-1} - S_t} = \frac{\mathcal{M}_{t-1}}{\mathcal{M}_{t-1}\Pi_t^{-1} + \theta S_{t-1}}$$

This demonstrates the irrelevance of randomizations in our analysis.